

CLASSIFICATION OF THE FINITE-DIMENSIONAL REAL DIVISION COMPOSITION ALGEBRAS HAVING A NON-ABELIAN DERIVATION ALGEBRA

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ABSTRACT. We classify the category of finite-dimensional real division composition algebras having a non-abelian Lie algebra of derivations. Our complete and explicit classification is largely achieved by introducing the concept of a quasi-description of a category, and using it to express the problem in terms of normal form problems for certain group actions on products of 3-spheres.

1. INTRODUCTION

Division algebras and composition algebras constitute two important classes of not necessarily associative algebras which are defined over any field. Over the real numbers, finite-dimensional algebras from either class exist exclusively in dimensions 1, 2, 4 and 8. For division algebras, this was proven in the classical work of Hopf [15], Bott–Milnor [5], and Kervaire [17]. For composition algebras, this is true over any field, and was proven by Kaplansky [16] in 1953. Despite their long history, the problem of classifying these algebras is still unsolved, even under the assumption of the ground field being \mathbb{R} and the dimension being finite.

In this study we consider finite-dimensional real algebras which are both division algebras and composition algebras. These algebras are characterized by the property that they are absolute valued, i.e. equipped with a multiplicative norm. The fact that these algebras only exist in dimensions 1, 2, 4 and 8 was proven by Albert [1] already in 1947. While a classification exists in dimension at most four, the case of dimension eight is far from fully understood, and has exhibited difficulties.

One way to approach these classes of algebras is by considering those objects which exhibit a high degree of symmetry, in the sense of having a large automorphism group. Examples are the algebras of the octonions, para-octonions, and the Okubo algebras, which, as the references indicate, have many interesting connections to various algebraic objects, as well as applications in mathematics and physics. For a real division algebra A , the automorphism group is a Lie group, its Lie algebra being the derivation algebra $\text{Der}(A)$, and A carries the structure of a $\text{Der}(A)$ -module. This allows for tools from representation theory to be used. The approach to finite-dimensional real division algebras via their derivation algebras was taken by Benkart and Osborn in [3] and [4], where they determined which Lie algebras may occur as such derivation algebras, and how the division algebras decompose as modules over their derivation algebras. For division algebras with large derivation algebras, they determined the multiplication tables as well. A similar

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study, using automorphism groups instead of derivation algebras, was conducted in [11]. Turning to finite-dimensional composition algebras over more general fields, several investigations were carried out by Elduque, Elduque–Myung and Elduque–Pérez, focusing mainly on algebras with large derivation algebras (see e.g. [12],[14]). In the extensive paper [19], Pérez gave an explicit account of all finite-dimensional division composition algebras with non-abelian derivation algebras over any field k of characteristic different from 2 and 3. Given a non-abelian Lie algebra L occurring as such a derivation algebra, he expressed all finite-dimensional division composition k -algebras A with $\text{Der}(A) = L$ as isotopes of unital composition algebras, and determined their decomposition into irreducible L -modules.

With the results from [19] as a starting point, we work toward classifying the finite-dimensional real division composition algebras having a non-abelian derivation algebra. The paper is organized as follows. After the necessary preliminaries, we recall the classification in the case of dimension at most 4. In Section 2 we review some general results on eight-dimensional real division composition algebras in preparation for our further study. From [19] we know that if A is an eight-dimensional division composition algebra over a field of characteristic not 2 or 3, having a non-abelian derivation algebra, and if $A = A_1 \oplus \cdots \oplus A_n$ is a decomposition of A into irreducible $\text{Der}(A)$ -modules, then the set $\{\dim A_i | 1 \leq i \leq n\}$ is invariant under isomorphism, and attains one of eight possible values. Accordingly, the category \mathcal{D} of eight-dimensional real division composition algebras having a non-abelian derivation algebra decomposes into eight blocks. In Section 3, we give a description (in the sense of Dieterich) of five of these, thus obtaining, in each case, an equivalence of categories from a group action groupoid to the block at hand.

The descriptions of Section 3 give precise information about the structure of the blocks, and extracting a classification out of them would be technical. Aiming at a more transparent approach, we introduce in Section 4 the notion of a quasi-description of a category. Like descriptions, a quasi-description of a category transfers the problem of classifying it to the normal form problem of a group action. Applying this to \mathcal{D} , we obtain two quasi-descriptions, together covering six of the eight blocks. In both quasi-descriptions, the group actions are induced by the action of the group $\text{Aut}(\mathbb{H}) \simeq SO_3$ on the set $\mathbb{S}^3 = \mathbb{S}(\mathbb{H})$ of quaternions of norm one. This gives a unified approach which also renders the classification problem quite feasible. In Section 5 we treat the remaining two blocks, achieving an explicit classification of each. Our final Section 6 is devoted to the solution of the normal form problem for the actions involved in the aforementioned quasi-descriptions, thereby completing the classification of \mathcal{D} .

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1.1. Preliminaries. By an *algebra* over a field k we understand a k -vector space A with a bilinear multiplication, written as juxtaposition, neither assumed to be associative or commutative, nor to admit a unity. A non-zero k -algebra A is called a *division algebra* if for each $a \in A \setminus \{0\}$, the operators $L_a = L_a^A$ and $R_a = R_a^A$ of left and right multiplication by a , respectively, are bijective. If A is a division algebra, then it has no zero divisors, and the converse is true if the dimension of A is finite. A k -algebra A is called a *composition algebra* if A is equipped with a

non-degenerate¹ quadratic form q that is multiplicative, i.e. such that

$$q(xy) = q(x)q(y)$$

for all $x, y \in A$. Finally, a non-zero real algebra A is called an *absolute valued algebra* if it is endowed with a multiplicative norm.

In the setting of finite-dimensional real algebras, the norm of an absolute valued algebra and the quadratic form of a composition algebra are each uniquely determined by the multiplication of the algebra. Thus we may speak of *the* norm of an absolute valued algebra and *the* quadratic form of a composition algebra. If a composition algebra A is moreover unital, it is equipped with an involution $J : A \rightarrow A$ fixing the unity and acting as -1 on its orthogonal complement. We will refer to this as the *standard involution on A* .

The following result expands on some of the remarks in the introduction.

Proposition 1.1. *Let A be a finite-dimensional, non-zero real algebra. Then the following statements are equivalent.*

- (i) *A is absolute valued.*
- (ii) *A is a division composition algebra.*
- (iii) *A is a composition algebra and the quadratic form q of A is positive definite.*
- (iv) *A is a composition algebra and the quadratic form q of A is definite.*

We call a quadratic form *definite* if it is positive definite or negative definite.

The above result is not hard to prove. For instance, (iii) is equivalent to (iv) since if $q(a) < 0$ for some $a \in A$, then $q(a^2) = q(a)^2 > 0$, whence q cannot be negative definite. The equivalence of the first three items is discussed in [8]. There, it is derived from the following important result from [1].

Proposition 1.2. *Every unital finite-dimensional absolute valued algebra is isomorphic to some $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. Any finite-dimensional absolute valued algebra is isomorphic to an orthogonal isotope $\mathbb{A}_{f,g}$ of some $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, i.e. $A = \mathbb{A}$ as a vector space, and the multiplication \cdot in A is given by*

$$x \cdot y = f(x)g(y)$$

for all $x, y \in A$, where f and g are linear orthogonal operators on A , and juxtaposition denotes multiplication in \mathbb{A} . Moreover, the norm in $\mathbb{A}_{f,g}$ coincides with the norm in \mathbb{A} .

We denote by \mathcal{A} the category of all finite-dimensional absolute valued algebras (hence, of all finite-dimensional real division composition algebras), where the morphisms are the non-zero algebra homomorphisms. It follows from the above result that the object class of \mathcal{A} is partitioned as

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_4 \cup \mathcal{A}_8,$$

where \mathcal{A}_d is the full subcategory of \mathcal{A} consisting of all d -dimensional objects. Since homomorphisms of finite-dimensional division algebras are injective, it follows that \mathcal{A}_d is a (not necessarily small) groupoid for each $d \in \{1, 2, 4, 8\}$. For $d > 1$, we have the following decomposition of \mathcal{A}_d , due to [9].

¹A quadratic form q on a k -vector space V , where $\text{char}(k) \neq 2$, is called *non-degenerate* if no $x \in V \setminus \{0\}$ is orthogonal to the entire space V with respect to the bilinear form $b_q(x, y) = q(x + y) - q(x) - q(y)$. In characteristic two the definition is more involved; see e.g. [18].

Proposition 1.3. *Let $A \in \mathcal{A}_d$ for some $d \in \{2, 4, 8\}$. For any $a, b \in A \setminus \{0\}$,*

$$\operatorname{sgn}(\det L_a) = \operatorname{sgn}(\det L_b) \quad \text{and} \quad \operatorname{sgn}(\det R_a) = \operatorname{sgn}(\det R_b).$$

The double sign of A is the pair $(\operatorname{sgn}(\det L_a), \operatorname{sgn}(\det R_a))$, $a \in A \setminus \{0\}$. Moreover, for each $d \in \{2, 4, 8\}$,

$$(1.1) \quad \mathcal{A}_d = \coprod_{(i,j) \in \mathbb{Z}_2^2} \mathcal{A}_d^{ij}$$

where \mathcal{A}_d^{ij} is the full subcategory of \mathcal{A}_d formed by all algebras having double sign $((-1)^i, (-1)^j)$.

Remark 1.4. If A is unital, then A has double sign $(+, +)$, and each orthogonal isotope $A_{f,g}$ thus has double sign $(\det g, \det f)$.

For each $d \in \{2, 4, 8\}$ and each subcategory \mathcal{B} of \mathcal{A}_d , we use the notation

$$\mathcal{B}^{ij} = \mathcal{B} \cap \mathcal{A}_d^{ij}.$$

We stress that the superscript belongs to \mathbb{Z}_2^2 . While it is more customary to use pairs of the signs $+$ and $-$, hence writing $\mathcal{B}^{\alpha\beta}$ where $\alpha = (-1)^i$ and $\beta = (-1)^j$, this will be notationally inconvenient for our purposes. Throughout this article, we will abuse notation by writing and treating the elements of \mathbb{Z}_2 as the natural numbers 0 and 1 with operations modulo 2.

In the study we are undertaking, we will use and generalize the concept of a *description* of a groupoid, due to Dieterich [10]. In this sense, a *description* of a groupoid \mathcal{C} is a quadruple $(G, M, \alpha, \mathcal{F})$ where G is a group, M is a set, $\alpha : G \times M \rightarrow M$ a group action, and $\mathcal{F} : {}_G M \rightarrow \mathcal{C}$ an equivalence of categories. The groupoid ${}_G M$ is defined by having M as object set, and for each $x, y \in M$, the morphism set

$${}_G M(x, y) = \{(g, x, y) | g \cdot x = y\}.$$

We denote the morphism (g, x, y) by g if the domain and codomain are clear from context. By constructing a description of a groupoid, one transfers the isomorphism problem for this groupoid to the normal form problem for the group action involved, and classifying the groupoid then amounts to finding a transversal for the orbits of the group action.

The groups appearing in the descriptions below are sometimes constructed as semi-direct products. We include the definition in order to fix notation.

Definition 1.5. Let G and H be groups and $\beta : H \times G \rightarrow G$ a group action such that for each $h \in H$, the map $\beta_h : g \mapsto h \cdot g$ is an automorphism of G . The *semi-direct product of G with H with respect to β* is the group $G \rtimes H = G \rtimes_{\beta} H$ with underlying set $G \times H$, and multiplication

$$(g, h)(g', h') = (g\beta_h(g'), hh').$$

1.2. Derivations. A *derivation* on an algebra A over a field k is a linear map $\delta : A \rightarrow A$ satisfying the Leibniz rule

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for any $x, y \in A$. If δ and δ' are two derivations of A , then so is the combination $r\delta + s\delta'$ for each $r, s \in k$ and the commutator $\delta\delta' - \delta'\delta$. The set of all derivations of A is a Lie algebra over k under the commutator, denoted $\operatorname{Der}(A)$. The algebra A becomes a $\operatorname{Der}(A)$ -module under the action $\delta \cdot a = \delta(a)$.

For each $d \in \{1, 2, 4, 8\}$, we will write \mathcal{D}_d for the full subcategory of \mathcal{A}_d whose object class consists of those algebras having a non-abelian derivation algebra. In dimension at most four, the following result captures the situation completely.

Proposition 1.6. (i) $\mathcal{D}_1 = \mathcal{D}_2 = \emptyset$ and \mathcal{D}_4 is classified by $\{\mathbb{H}_{K^j, K^i} \mid (i, j) \in \mathbb{Z}_2^2\}$, where K is the standard involution $x \mapsto \bar{x}$ in \mathbb{H} .
(ii) For each $(i, j) \in \mathbb{Z}_2^2$, $L = \text{Der}(\mathbb{H}_{K^j, K^i})$ is of type \mathfrak{su}_2 , and $\mathbb{H}_{K^j, K^i} = \mathbb{R}1 \oplus 1^\perp$ as an L -module, both direct summands being irreducible.
(iii) The automorphism group of \mathbb{H}_{K^j, K^i} is

$$SO_3 = SO(1^\perp) = \{\kappa_q \mid q \in \mathbb{S}^3 = \mathbb{S}(\mathbb{H})\}.$$

(iv) For each $(i, j) \in \mathbb{Z}_2^2$, a description of \mathcal{D}_4^{ij} is given by $(SO_3, \{*\}, \alpha, \mathcal{E}^{ij})$, where α is the trivial action of SO_3 on the singleton set $\{*\}$, and

$$\mathcal{E}^{ij} : SO_3 \{*\} \rightarrow \mathcal{D}_4^{ij}$$

acts on objects by $\{*\} \mapsto \mathbb{H}_{K^j, K^i}$, and on morphisms as identity.

The first two items are due to [14], the third is well-known, and the last is a direct consequence of these. As above, we will henceforth identify SO_3 with $\{\kappa_q \mid q \in \mathbb{S}(\mathbb{H})\}$, where $\kappa_q : \mathbb{H} \rightarrow \mathbb{H}$ is defined by $x \mapsto qx\bar{q}$.

Remark 1.7. In the above proposition we saw that each \mathbb{H}_{K^j, K^i} decomposes into irreducible submodules, in other words, is *completely reducible* as a module over its derivation algebra. This is true for any finite-dimensional division composition algebra over any field of characteristic not 2 or 3, as was observed in [19].

For the remainder of the paper we will be preoccupied with the full subcategory $\mathcal{D} = \mathcal{D}_8 \subset \mathcal{A}_8$. We define, for each partition Π of the natural number 8, the full subcategory \mathcal{D}_Π of \mathcal{D} , in which the objects are all $A \in \mathcal{D}$ such that Π the partition of 8 given by the dimensions of the irreducible components of A as a $\text{Der}(A)$ -module.

Example 1.8. For the algebra \mathbb{O} we know that $\text{Der}(\mathbb{O})$ is a Lie algebra of type \mathfrak{g}_2 , and that as a \mathfrak{g}_2 -module, \mathbb{O} is the direct sum of a (trivial) one-dimensional module, and an irreducible seven-dimensional module. Thus $\mathbb{O} \in \mathcal{D}_{\{1,7\}}$. In fact, $\{\mathbb{O}\}$ exhausts $\mathcal{D}_{\{1,7\}}^{00}$ up to isomorphism.

In the sequel we shall omit the brackets in the subscript Π , except in the case where $\Pi = \{8\}$, to avoid confusion with $\mathcal{D}_8 = \mathcal{D}$.

From [19] we have the following decomposition of \mathcal{D} .

Proposition 1.9. *The category \mathcal{D} decomposes as the coproduct*

$$\mathcal{D}_{1,7} \amalg \mathcal{D}_{\{8\}} \amalg \mathcal{D}_{1,1,6} \amalg \mathcal{D}_{1,3,4} \amalg \mathcal{D}_{1,1,2,4} \amalg \mathcal{D}_{1,1,1,1,4} \amalg \mathcal{D}_{3,5} \amalg \mathcal{D}_{1,1,3,3}.$$

For each $A \in \mathcal{D}$, $\text{Der}(A)$ is of type \mathfrak{g}_2 if $A \in \mathcal{D}_{1,7}$, \mathfrak{su}_3 if $A \in \mathcal{D}_{\{8\}} \amalg \mathcal{D}_{1,1,6}$, $\mathfrak{su}_2 \times \mathfrak{su}_2$ or $\mathfrak{su}_2 \times \mathfrak{a}$ if $A \in \mathcal{D}_{1,3,4}$, and $\mathfrak{su}_2 \times \mathfrak{a}$ otherwise, where \mathfrak{a} is an abelian Lie algebra with $\dim \mathfrak{a} \leq 1$.

We will write $\mathcal{D}_{1,3,4}^s$ and $\mathcal{D}_{1,3,4}^a$ for the full subcategory of $\mathcal{D}_{1,3,4}$ in which the objects have a derivation algebra of type $\mathfrak{su}_2 \times \mathfrak{su}_2$ and $\mathfrak{su}_2 \times \mathfrak{a}$, respectively. Thus $\mathcal{D}_{1,3,4} = \mathcal{D}_{1,3,4}^s \amalg \mathcal{D}_{1,3,4}^a$.

For each $A \in \mathcal{D}$, the sum of all one-dimensional $\text{Der}(A)$ -submodules of A is the trivial submodule

$$A_0 = \{a \in A \mid \forall \delta \in \text{Der}(A) : \delta(a) = 0\}.$$

The following result is well-known.

Lemma 1.10. *Let $A \in \mathcal{D}$. Then A_0 is a proper composition subalgebra of A . Hence $\dim A_0 \in \{0, 1, 2, 4\}$.*

Remark 1.11. Let $A, B \in \mathcal{D}$ and assume that $\phi : A \rightarrow B$ is an isomorphism. Then for each $\delta \in \text{Der}(A)$ we have $\phi\delta\phi^{-1} \in \text{Der}(B)$. It follows that $\phi(A_0) = B_0$. Moreover, if $C \subseteq A$ is an irreducible $\text{Der}(A)$ -submodule, then $\phi(C)$ is an irreducible $\text{Der}(B)$ -submodule. From the fact (see [14]) that $\delta(x) \perp x$ for each $x \in A$, we see that the orthogonal projection of C on any submodule of A is itself a submodule. From this it follows that if $A \in \mathcal{D}_\Pi$, and the natural number $d = \dim C$ occurs precisely once in the partition Π , then C is the unique d -dimensional submodule of A , and $\phi(C)$ is the unique d -dimensional submodule of B .

We will refer to the results on derivation algebras of division composition algebras obtained in [19] along the way as we need them. Those results which are not necessary for our arguments will be left out in order to keep this article within reasonable limits. The reader is referred to [19] for these results and their proofs.

2. EIGHT-DIMENSIONAL REAL DIVISION COMPOSITION ALGEBRAS

By Proposition 1.2, each $A \in \mathcal{A}_8$ is isomorphic to $\mathbb{O}_{f,g}$ for some $f, g \in O(\mathbb{O})$. In this section, we will recall some results on when two such algebras are isomorphic. To begin with, we will review some basic properties of the automorphism group of \mathbb{O} . This is a 14-dimensional compact exceptional Lie group of type G_2 , which we will denote by G_2 for notational simplicity. Identifying the groups O_8 and SO_8 with $O(\mathbb{O})$ and $SO(\mathbb{O})$, respectively, we have $G_2 < SO_8$. One way to visualize G_2 is by using Cayley triples. A *Cayley triple* is an orthonormal triple (u, v, z) , where $u, v, z \in \mathbb{O}$ are purely imaginary (i.e. orthogonal to 1) and satisfy $z \perp uv$. Any permutation of a Cayley triple is a Cayley triple. Given a Cayley triple (u, v, z) , the element u generates the two-dimensional subalgebra $\langle 1, u \rangle \simeq \mathbb{C}$, and $\{u, v\}$ generates the four-dimensional subalgebra $\langle 1, u, v, uv \rangle \simeq \mathbb{H}$.² It is known that the group G_2 acts transitively on the set of all Cayley triples, and that for any two such triples there is a unique $\phi \in G_2$ mapping one to the other. Thus by fixing a Cayley triple (u, v, z) once and for all, the elements of G_2 correspond bijectively to the set of all Cayley triples via $\phi \mapsto (\phi(u), \phi(v), \phi(z))$.

For the remainder of this article, we fix a Cayley triple (u, v, z) in \mathbb{O} . In view of the above, we will often identify the subalgebras $\langle 1, u \rangle$ and $\langle 1, u, v, uv \rangle$ of \mathbb{O} with \mathbb{C} and \mathbb{H} , respectively. Thus \mathbb{C} and \mathbb{H} become subspaces of $\mathbb{O}_{f,g}$ for each $f, g \in O_8$. Moreover, we denote $\mathbb{S}(\mathbb{C})$ by \mathbb{S}^1 and $\mathbb{S}(\mathbb{H})$ by \mathbb{S}^3 . Finally, we denote the standard involution $x \mapsto \bar{x}$ on \mathbb{O} (and thus on \mathbb{C} and \mathbb{H} as well) by K .

Proposition 1.2 and the above discussion imply the following well-known fact.

Lemma 2.1. *Let $A \in \mathcal{A}_8$ be unital, $B \subset A$ a four-dimensional unital subalgebra, and $C \subset B$ a two-dimensional unital subalgebra. Then there is an isomorphism $\phi : A \rightarrow \mathbb{O}$ such that $\phi(B) = \langle 1, u, v, uv \rangle$ and $\phi(C) = \langle 1, u \rangle$.*

We will often need to establish isomorphisms between isotopes of arbitrary unital algebras in \mathcal{A}_8 and isotopes of \mathbb{O} . For this, we use the following result from [6].

²If V is a vectorspace and $u_1, \dots, u_n \in V$, then $\langle u_1, \dots, u_n \rangle$ always denotes their linear span.

Lemma 2.2. *Let $A, B \in \mathcal{A}_8$ and assume there is an isomorphism $\phi : A \rightarrow B$. Then for each $f, g \in O_8$, $\phi : A_{f,g} \rightarrow B_{\phi f \phi^{-1}, \phi g \phi^{-1}}$ is an isomorphism.*

This was used in [6] to prove the following isomorphism condition.

Proposition 2.3. *Let $f, g, f', g' \in O_8$. Then $\phi : \mathbb{O}_{f,g} \rightarrow \mathbb{O}_{f',g'}$ is an isomorphism if and only if $\phi \in SO_8$ and*

$$(2.1) \quad \phi_1 f \phi^{-1} = f' \quad \text{and} \quad \phi_2 g \phi^{-1} = g',$$

where (ϕ_1, ϕ_2) is a triality pair of ϕ .

A triality pair of $\phi \in SO_8$ is a pair $(\phi_1, \phi_2) \in SO_8^2$ satisfying $\phi(xy) = \phi_1(x)\phi_2(y)$ for any $x, y \in \mathbb{O}$. The Principle of triality, due to E. Cartan [7], asserts that for each $\phi \in SO_8$, a triality pair exists and is unique up to sign. It follows from the definition that if (ϕ_1, ϕ_2) is a triality pair of ϕ , then

$$(2.2) \quad \phi_1 = R_{\phi_2(1)}^{\mathbb{O}} \phi \quad \text{and} \quad \phi_2 = L_{\phi_1(1)}^{\mathbb{O}} \phi.$$

Using the above result, we obtained in [2] a description of the category \mathcal{A}_8 . This description is given by the quadruple

$$(SO_8, [O_8 \times O_8], \tau, \mathcal{T}),$$

where $[O_8 \times O_8]$ is the quotient $(O_8 \times O_8)/\{\pm(\text{Id}, \text{Id})\}$, and τ is the *triality action*

$$SO_8 \times [O_8 \times O_8] \rightarrow [O_8 \times O_8], \quad (\phi, [f, g]) \mapsto [\phi_1 f \phi^{-1}, \phi_2 g \phi^{-1}],$$

where (ϕ_1, ϕ_2) is any triality pair of ϕ . The functor

$$\mathcal{T} : SO_8[O_8 \times O_8] \rightarrow \mathcal{A}_8$$

acts on objects by $[f, g] \mapsto \mathbb{O}_{f,g}$, and on morphisms by $\phi \mapsto \phi$.

By definition of an automorphism, $\phi \in SO_8$ belongs to G_2 if and only if (ϕ, ϕ) is a triality pair of ϕ . (By (2.2), this is equivalent to ϕ having a triality pair (ϕ_1, ϕ_2) with $\phi(1) = \phi_1(1) = \phi_2(1) = 1$.) Thus if one restricts the triality action to $G_2 < SO_8$, one obtains the action

$$\gamma : G_2 \times [O_8 \times O_8] \rightarrow [O_8 \times O_8], \quad (\phi, [f, g]) \mapsto [\phi f \phi^{-1}, \phi g \phi^{-1}].$$

However, the inclusion functor

$$G_2[O_8 \times O_8] \rightarrow SO_8[O_8 \times O_8],$$

is obviously not full, and therefore not all morphisms in \mathcal{A}_8 belong to G_2 . The following lemma presents one condition under which all morphisms between two algebras $\mathbb{O}_{f,g}$ and $\mathbb{O}_{f',g'}$ belong to G_2 . As it turns out, it will often be the case that the algebras we consider here satisfy this condition.

Lemma 2.4. *Let $f, g, f', g' \in O_8$. If there exists a subspace $\{0\} \neq U \subseteq \mathbb{R}^8$ such that the restriction of each of f, g, f' and g' to U is the identity, then $\phi : \mathbb{O}_{f,g} \rightarrow \mathbb{O}_{f',g'}$ is an isomorphism with $\phi(U) = U$ if and only if $\phi \in G_2$, $\phi(U) = U$ and $(f', g') = (\phi f \phi^{-1}, \phi g \phi^{-1})$.*

Proof. If $\phi \in G_2$ and $(f', g') = (\phi f \phi^{-1}, \phi g \phi^{-1})$, then $(f', g') = (\phi_1 f \phi^{-1}, \phi_2 g \phi^{-1})$ for the triality pair $(\phi_1, \phi_2) = (\phi, \phi)$ of ϕ , and by Proposition 2.3, ϕ is an isomorphism. If, conversely, $\phi : \mathbb{O}_{f,g} \rightarrow \mathbb{O}_{f',g'}$ is an isomorphism, then for some triality pair (ϕ_1, ϕ_2) of ϕ ,

$$R_{\phi_2(1)}^{\mathbb{O}} \phi f \phi^{-1} = \phi_1 f \phi^{-1} = f' \quad \text{and} \quad L_{\phi_1(1)}^{\mathbb{O}} \phi g \phi^{-1} = \phi_2 g \phi^{-1} = g'.$$

Now $\phi f \phi^{-1}$ and $\phi g \phi^{-1}$ act as identity on U , whence applying both sides of the two above equalities to any $u \in U \setminus 0$ one obtains the equalities

$$u = \phi_2(1)u \quad \text{and} \quad u = \phi_1(1)u$$

in \mathbb{O} . Since \mathbb{O} is a unital division algebra we get $\phi_1(1) = \phi_2(1) = 1$, whence $\phi(1) = 1$, and altogether $\phi \in G_2$. Thus $[f', g'] = [\phi f \phi^{-1}, \phi g \phi^{-1}]$, and then $(f', g') = (\phi f \phi^{-1}, \phi g \phi^{-1})$ since e.g. $\phi f \phi^{-1}$ and f' must have the same sign, by the condition on their restrictions to U . \square

Remark 2.5. Choosing $U = \langle 1 \rangle$ in the lemma, we deduce the fact obtained in [6] that if $f(1) = g(1) = f'(1) = g'(1) = 1$, then $\phi : \mathbb{O}_{f,g} \rightarrow \mathbb{O}_{f',g'}$ is an isomorphism fixing 1 if and only if $\phi \in G_2$ and $(f', g') = (\phi f \phi^{-1}, \phi g \phi^{-1})$.

3. DESCRIPTIONS IN DIMENSION EIGHT

In this section, we will construct a description of the subcategory \mathcal{B}^{ij} of \mathcal{D} for each $(i, j) \in \mathbb{Z}_2^2$, and each \mathcal{B} among

$$\mathcal{D}_{1,7}, \quad \mathcal{D}_{1,1,6}, \quad \mathcal{D}_{1,3,4}, \quad \mathcal{D}_{1,1,2,4} \quad \text{and} \quad \mathcal{D}_{1,1,1,1,4}.$$

The subcategory $\mathcal{D}_{\{8\}}$ has been studied in several papers, giving a classification which we quote below. We will return to all these subcategories in Section 4, using the approach via quasi-descriptions. The reader may thus see this section as a means of paving the road for the next. The remaining two subcategories $\mathcal{D}_{3,5}$ and $\mathcal{D}_{1,1,3,3}$ will be treated in Section 5.

3.1. The Subcategory $\mathcal{D}_{1,7}$. Let $A \in \mathcal{A}_8$. By Proposition 1.9, $A \in \mathcal{D}_{1,7}$ if and only if $\text{Der}(A)$ is of type \mathfrak{g}_2 . By [14], this holds if and only if A is *standard*, that is, $A \simeq \mathbb{O}_{K^j, K^i}$ for some $(i, j) \in \mathbb{Z}_2^2$. For each of these isotopes, the trivial submodule is the subspace $\langle 1 \rangle$. The fact that these four isotopes of \mathbb{O} are pairwise non-isomorphic is well-known (and follows from e.g. Proposition 1.3), as is the fact that their automorphism groups coincide with the automorphism group of \mathbb{O} . From these observations, the following descriptions of the double sign components of $\mathcal{D}_{1,7}$ are immediately obtained.

Proposition 3.1. *The group G_2 acts trivially on the singleton set $\{*\}$, and for each $(i, j) \in \mathbb{Z}_2^2$, the map*

$$G_2 \{*\} \rightarrow \mathcal{D}_{1,7}^{ij}$$

acting on objects by $\{\} \mapsto \mathbb{O}_{K^j, K^i}$, and on morphisms by $\phi \mapsto \phi$, is an equivalence of categories.*

3.2. The Subcategories $\mathcal{D}_{\{8\}}$ and $\mathcal{D}_{1,1,6}$. We begin with the following characterization of $\mathcal{D}_{\{8\}}$ due to [14].

Lemma 3.2. *Let $A \in \mathcal{A}_8$. Then $A \in \mathcal{D}_{\{8\}}$ if and only if A is an Okubo algebra.*

Okubo algebras have been treated extensively in the literature. For a definition, see e.g [12]. The below construction is readily derived from [19].

Example 3.3. Define the algebra

$$P^{11} = \mathbb{O}_{K^\tau, K^{\tau^{-1}}},$$

where $\tau = \tau_{(\sqrt{3}u-1)/2} \in G_2$ is defined by $(u, v, z) \mapsto (u, v, z(\sqrt{3}u-1)/2)$. Then P^{11} is a division Okubo algebra with double sign $(-, -)$.

In [12] it was shown that there is a unique isomorphism class of real division Okubo algebras. This implies that $\mathcal{D}_{\{8\}} = \mathcal{D}_{\{8\}}^{11}$ is classified by $\{P^{11}\}$.

For $\mathcal{D}_{1,1,6}$, a dense subset together with an isomorphism condition were given in [14] as follows.

Lemma 3.4. *Let $A \in \mathcal{A}_8$. Then $A \in \mathcal{D}_{1,1,6}$ if and only if $A \simeq C_{f,g}$, where $C \in \mathcal{A}_8$ is unital with standard involution J , and where for some two-dimensional unital subalgebra B of C , the pair $(f, g) \in O(C)^2$ satisfies*

- (i) $f(B) = g(B) = B$ and $f|_{B^\perp} = g|_{B^\perp} = \text{Id}|_{B^\perp}$, and
- (ii) $f, g \notin \{\text{Id}, -J\}$.

Moreover, if $(f, g), (f', g') \in O(C)^2$ satisfy (i) and (ii) for subalgebras B and B' , respectively, then $\phi : C_{f,g} \rightarrow C_{f',g'}$ is an isomorphism if and only if $\phi \in \text{Aut}(C)$ with $\phi f \phi^{-1} = f'$ and $\phi g \phi^{-1} = g'$. Finally if $B = B'$, then $C_{f,g} \simeq C_{f',g'}$ if and only if

$$J^k f J^k = f' \quad \text{and} \quad J^l g J^l = g'$$

for some $(k, l) \in \mathbb{Z}_2^2$.

It was shown in [14] that if $A = C_{f,g}$ is as in the lemma, then $A_0 = B$.

Using this, we next give a description of $\mathcal{D}_{1,1,6}^{ij}$ for each $(i, j) \in \mathbb{Z}_2^2$. For this, we consider the subgroup

$$G_2^{(u)} = \{\phi \in G_2 \mid \phi(u) = \pm u\}$$

of G_2 . It is known that the subgroup of all $\phi \in G_2$ fixing u is isomorphic to SU_3 . Identifying SU_3 with this subgroup, we see that $G_2^{(u)}$ is isomorphic to the semi-direct product $SU_3 \rtimes \mathbb{Z}_2$ with respect to the action

$$\mathbb{Z}_2 \times SU_3 \rightarrow SU_3 \quad (\varepsilon, \rho) \mapsto \widehat{\varepsilon} \rho \widehat{\varepsilon},$$

where $\widehat{\varepsilon} \in G_2$ is defined by $(u, v, z) \mapsto ((-1)^\varepsilon u, v, z)$. The isomorphism takes each $(\rho, \varepsilon) \in SU_3 \rtimes \mathbb{Z}_2$ to $\rho \widehat{\varepsilon} \in G_2^{(u)}$. The description is given by the following result.

Proposition 3.5. *Let $(i, j) \in \mathbb{Z}_2^2$. The group $SU_3 \rtimes \mathbb{Z}_2$ acts on*

$$(\mathbb{S}^1 \times \mathbb{S}^1)_{ij} = (\mathbb{S}^1 \times \mathbb{S}^1) \setminus \{((-1)^j, (-1)^i)\}$$

by

$$(3.1) \quad (\rho, \varepsilon) \cdot (a, b) = (K^\varepsilon(a), K^\varepsilon(b)).$$

The map

$$\mathcal{F}^{ij} : SU_3 \rtimes \mathbb{Z}_2 (\mathbb{S}^1 \times \mathbb{S}^1)_{ij} \rightarrow \mathcal{D}_{1,1,6}^{ij}$$

acting on objects by $(a, b) \mapsto \mathbb{O}_{\lambda_a^{(j)}, \lambda_b^{(i)}}$, and on morphisms by $(\rho, \varepsilon) \mapsto \rho \widehat{\varepsilon}$, is an equivalence of categories.

For each $t \in \mathbb{S}^1$ and $k \in \mathbb{Z}_2$, the map $\lambda_t^{(k)} : \mathbb{O} \rightarrow \mathbb{O}$ is defined by

$$\lambda_t^{(k)}(x) = \begin{cases} tK^k(x) & \text{if } x \in \mathbb{C}, \\ x & \text{if } x \in \mathbb{C}^\perp. \end{cases}$$

Thus with $C = \mathbb{O}$, $B = \mathbb{C}$ and $(f, g) = (\lambda_a^{(j)}, \lambda_b^{(i)})$, the algebra $C_{f,g}$ satisfies items (i) and (ii) of Lemma 3.4.

Proof. Let $(i, j) \in \mathbb{Z}_2^2$. Equation (3.1) defines an action of $SU_3 \rtimes \mathbb{Z}_2$ on $\mathbb{S}^1 \times \mathbb{S}^1$ which fixes $\{((-1)^j, (-1)^i)\}$ and thus induces the desired action on its complement. The map \mathcal{F}^{ij} is well-defined on objects by Lemma 3.4, since $\lambda_t^{(k)} \in \{\text{Id}, -K\}$ if and only if $t = (-1)^k$. To show that it is well-defined on morphisms, we first note that it maps identity morphisms to identity morphisms. Further, assume that $(\rho, \varepsilon) \cdot (a, b) = (c, d)$ for some $(\rho, \varepsilon) \in SU_3 \rtimes \mathbb{Z}_2$ and $(a, b), (c, d) \in \mathbb{S}^1 \times \mathbb{S}^1$. Then for each $x \in \mathbb{C}$,

$$(3.2) \quad \rho \widehat{\varepsilon} \lambda_a^{(j)} (\rho \widehat{\varepsilon})^{-1} (x) = K^\varepsilon (a K^j K^\varepsilon (x)) = K^\varepsilon (a) K^j (x) = \lambda_{K^\varepsilon(a)}^{(j)} (x),$$

and on \mathbb{C}^\perp both $\rho \widehat{\varepsilon} \lambda_a^{(j)} (\rho \widehat{\varepsilon})^{-1}$ and $\lambda_c^{(j)}$ act as identity. Thus $\rho \widehat{\varepsilon} \lambda_a^{(j)} (\rho \widehat{\varepsilon})^{-1} = \lambda_c^{(j)}$ and, analogously, $\rho \widehat{\varepsilon} \lambda_b^{(i)} (\rho \widehat{\varepsilon})^{-1} = \lambda_d^{(i)}$. Since $\rho \widehat{\varepsilon} \in G_2$, Lemma 3.4 implies that

$$\rho \widehat{\varepsilon} : \mathbb{O}_{\lambda_a^{(j)}, \lambda_b^{(i)}} \rightarrow \mathbb{O}_{\lambda_c^{(j)}, \lambda_d^{(i)}},$$

whence \mathcal{F}^{ij} is well-defined on morphisms. Functoriality (i.e. the property that the map respects composition of morphisms) is due to the fact that $(\rho, \varepsilon) \mapsto \rho \widehat{\varepsilon}$ defines a group homomorphism from $SU_3 \rtimes \mathbb{Z}_2$ to $G_2^{(u)}$.

To prove denseness, let $A \in \mathcal{D}_{1,1,6}^{ij}$. Then Lemma 3.4 implies that $A \simeq C_{f,g}$ for a unital algebra $C \in \mathcal{A}_8$ and $f, g \in O(C)$ that map a two-dimensional unital subalgebra $B \subset C$ to itself and are the identity on B^\perp . Lemma 2.1 then implies that there is an isomorphism $\phi : C \rightarrow \mathbb{O}$ mapping B to \mathbb{C} . Thus by Lemma 2.2 we have $A \simeq \mathbb{O}_{f',g'}$ with $f' = \phi f \phi^{-1}$ and $g' = \phi g \phi^{-1}$ mapping \mathbb{C} to itself and being the identity on \mathbb{C}^\perp . By double sign considerations, $\det f' = j$ and $\det g' = i$. Recall that an orthogonal map on \mathbb{C} with determinant k is of the form $x \mapsto t K^k(x)$ for some $t \in \mathbb{S}^1$. Hence there exist $a, b \in \mathbb{S}^1$ such that $f' = \lambda_a^{(j)}$ and $g' = \lambda_b^{(i)}$. Finally condition (ii) from Lemma 3.4 ensures that $(a, b) \neq ((-1)^j, (-1)^i)$. Thus A is isomorphic to $\mathcal{F}^{ij}(a, b)$, proving denseness.

The functor is faithful since if $\rho \widehat{\varepsilon} = \sigma \widehat{\eta}$, then applying both sides to the Cayley triple (u, v, z) we have $((-1)^\varepsilon u, \rho(v), \rho(z)) = ((-1)^\eta u, \sigma(v), \sigma(z))$. Thus clearly $\varepsilon = \eta$, and $\rho = \sigma$ as both maps fix u . To prove fullness, assume that

$$\phi : A = \mathbb{O}_{\lambda_a^{(j)}, \lambda_b^{(i)}} \rightarrow \mathbb{O}_{\lambda_c^{(j)}, \lambda_d^{(i)}} = B.$$

By Remark 1.11, ϕ maps the trivial $\text{Der}(A)$ -submodule $\mathbb{C} \subset A$ to the trivial $\text{Der}(B)$ -submodule $\mathbb{C} \subset B$, whence $\phi(\mathbb{C}^\perp) = \mathbb{C}^\perp$, and Lemma 2.4 applies, giving that $\phi \in G_2^{(u)}$ and satisfies

$$\phi \lambda_a^{(j)} \phi^{-1} = \lambda_c^{(j)} \quad \text{and} \quad \phi \lambda_b^{(i)} \phi^{-1} = \lambda_d^{(i)}.$$

The isomorphism $SU_3 \rtimes \mathbb{Z}_2 \rightarrow G_2^{(u)}$ established above implies that $\phi = \rho \widehat{\varepsilon}$ for some $(\rho, \varepsilon) \in SU_3 \rtimes \mathbb{Z}_2$, and from (3.2) we conclude that $\lambda_{K^\varepsilon(a)}^{(j)} = \lambda_c^{(j)}$. Applying both sides to 1 gives $K^\varepsilon(a) = c$, and by analogy we get $K^\varepsilon(b) = d$. Altogether $(\rho, \varepsilon) \cdot (a, b) = (c, d)$ and the functor is full, which completes the proof. \square

3.3. The Subcategory $\mathcal{D}_{1,3,4}$. Following [19] we define, for each $p \in \mathbb{S}^3 \subset \mathbb{H} \subset \mathbb{O}$, the map $\tau_p \in G_2$ by

$$\tau_p : (u, v, z) \mapsto (u, v, zp).$$

Moreover we define, for each $q \in \mathbb{S}^3$, the map $\widehat{\kappa}_q \in G_2$ by

$$\widehat{\kappa}_q : (u, v, z) \mapsto (qu\overline{q}, qv\overline{q}, z),$$

which is well-defined as \mathbb{H} is an associative subalgebra of \mathbb{O} . Recall that we have defined $\kappa_q : \mathbb{H} \rightarrow \mathbb{H}, x \mapsto qx\bar{q}$ for each $q \in \mathbb{S}^3$, and κ_q is the restriction of $\hat{\kappa}_q$ to $\mathbb{H} \subset \mathbb{O}$. Observe however that $\hat{\kappa}_q$ does not act as identity on $\mathbb{H}^\perp = \mathbb{H}z$; indeed $\hat{\kappa}_q(xz) = \kappa_q(x)z$ for each $x \in \mathbb{H}$.

These maps are useful to characterize those automorphisms of \mathbb{O} which fix \mathbb{H} , either pointwise or as a subalgebra. Thus we introduce the notation

$$G_2^{\mathbb{H}} = \{\phi \in G_2 \mid \phi(\mathbb{H}) = \mathbb{H}\} \quad \text{and} \quad G_2^{(\mathbb{H})} = \{\phi \in G_2^{\mathbb{H}} \mid \phi|_{\mathbb{H}} = \text{Id}|_{\mathbb{H}}\}.$$

It is well-known (see e.g. [20]) that $G_2^{\mathbb{H}} \simeq SO_4$, while it was shown in [19] that $G_2^{(\mathbb{H})} = \{\tau_p \mid p \in \mathbb{S}^3\}$. The following lemma encompasses these results and is adapted to suit our current formalism. Recall that we identify SO_3 with $\{\kappa_q \mid q \in \mathbb{S}^3\}$, and consider the semi-direct product $\mathbb{S}^3 \rtimes SO_3$ with respect to the action of SO_3 on \mathbb{S}^3 given by $\kappa_q \cdot p = \kappa_q(p)$.

Lemma 3.6. *The map*

$$\Delta : \mathbb{S}^3 \rtimes SO_3 \rightarrow G_2^{\mathbb{H}} \quad (p, \kappa_q) \mapsto \tau_{\bar{p}} \hat{\kappa}_q$$

is an isomorphism of groups, inducing an isomorphism $\mathbb{S}^3 \rightarrow G_2^{(\mathbb{H})}$, $p \mapsto \tau_{\bar{p}}$.

Proof. To prove that Δ is a group homomorphism we must show that for any $p, q, p', q' \in \mathbb{S}^3$, $\tau_{\bar{p}} \hat{\kappa}_q \tau_{\bar{p}'} \hat{\kappa}_{q'}$ equals

$$\Delta((p, \kappa_q)(p', \kappa_{q'})) = \tau_{\overline{pqp'q}} \hat{\kappa}_{qq'}.$$

Indeed, both map each $x \in \mathbb{H}$ to $\kappa_{qq'}(x)$, while

$$\tau_{\bar{p}} \hat{\kappa}_q \tau_{\bar{p}'} \hat{\kappa}_{q'}(z) = \tau_{\bar{p}} \hat{\kappa}_q(z \bar{p}') = \tau_{\bar{p}}(z(q \bar{p}' \bar{q})) = (z \bar{p})(q \bar{p}' \bar{q}) = z(\overline{pqp'q}) = \tau_{\overline{pqp'q}} \hat{\kappa}_{qq'}(z),$$

where we have used that

$$(3.3) \quad \forall x, y \in \mathbb{H}, (zx)y = z(yx),$$

which can be verified from a multiplication table of \mathbb{O} .

The map is injective since if $\tau_{\bar{p}} \hat{\kappa}_q = \tau_{\bar{p}'} \hat{\kappa}_{q'}$, then restricting to \mathbb{H} we get $\kappa_q = \kappa_{q'}$, while $z\bar{p} = z\bar{p}'$ implies that $p = p'$. To prove surjectivity, assume that $\rho \in G_2$ satisfies $\rho(\mathbb{H}) = \mathbb{H}$. Then $\rho|_{\mathbb{H}}$ is an automorphism of the algebra \mathbb{H} , and $\rho(z) \perp \mathbb{H}$. Then there exists $q \in \mathbb{S}^3$ and $z' \in \mathbb{S}(\mathbb{H}^\perp)$ such that ρ maps (u, v, z) to $(qu\bar{q}, qv\bar{q}, z')$. Then $z' = z\bar{p}$ with $p = \bar{z}'z \in \mathbb{H}$, and $\rho = \Delta(p, \kappa_q)$. The statement on the induced map holds since $\tau_{\bar{p}} \hat{\kappa}_q$ fixes \mathbb{H} pointwise if and only if $\kappa_q = \text{Id}|_{\mathbb{H}}$, i.e. if and only if $\tau_{\bar{p}} \hat{\kappa}_q \in \Delta(\mathbb{S}^3 \rtimes \{1\})$. \square

Remark 3.7. The above lemma and remarks imply the fact that $\mathbb{S}^3 \rtimes SO_3 \simeq SO_4$. This isomorphism can also be established directly. Identifying SO_4 with $SO(\mathbb{H})$, it is a classical result that each $\phi \in SO_4$ is of the form $x \mapsto axb$ for some $a, b \in \mathbb{S}^3$. One can then check that an isomorphism $\mathbb{S}^3 \rtimes SO_3 \rightarrow SO_4$ is obtained by mapping (p, κ_q) to the map $x \mapsto p\kappa_q(x)$.

Consider now the category $\mathcal{D}_{1,3,4}$, an exhaustive list of which is given in [19]. This is refined in the following result.

Lemma 3.8. *Let $(i, j) \in \mathbb{Z}_2^2$ and $A \in \mathcal{D}_{1,3,4}^{ij}$. Then $A \simeq \mathbb{O}_{K^j\tau_a, K^i\tau_b}$ for some $(a, b) \in (\mathbb{S}^3 \times \mathbb{S}^3)_{ij}$. Conversely, $\mathbb{O}_{K^j\tau_a, K^i\tau_b} \in \mathcal{D}_{1,3,4}^{ij}$ for each $(i, j) \in \mathbb{Z}_2^2$ and $(a, b) \in (\mathbb{S}^3 \times \mathbb{S}^3)_{ij}$.*

Here,

$$(\mathbb{S}^3 \times \mathbb{S}^3)_{ij} = \begin{cases} (\mathbb{S}^3 \times \mathbb{S}^3) \setminus \{(a, a^2) | a^2 + a + 1 = 0 \vee a = 1\} & \text{if } (i, j) = (1, 1), \\ (\mathbb{S}^3 \times \mathbb{S}^3) \setminus \{(1, 1)\} & \text{otherwise.} \end{cases}$$

Proof. Let $A \in \mathcal{D}_{1,3,4}^{ij}$. By [19] (Proposition 38), there exists $C \in \mathcal{A}_8$ with unity e , a fixed quaternion subalgebra Q and a fixed $w \in \mathbb{S}(Q^\perp)$ such that

$$A \simeq C_{J^l \tau'_x, J^k \tau'_y},$$

for some $(k, l) \in \mathbb{Z}_2^2$ and $x, y \in \mathbb{S}(Q)$, where J is the standard involution on C and τ'_x (resp. τ'_y) is the unique automorphism of C fixing Q pointwise and mapping w to wx (resp. wy). Then by Proposition 1.3 we necessarily have $(k, l) = (i, j)$. By Lemma 2.1 there is an isomorphism

$$\phi : C \rightarrow \mathbb{O}$$

mapping e to 1, Q to \mathbb{H} , and w to z . Lemma 2.2 then implies that we have the isomorphism

$$\phi : C_{J^j \tau'_x, J^i \tau'_y} \rightarrow \mathbb{O}_{\phi J^j \tau'_x \phi^{-1}, \phi J^i \tau'_y \phi^{-1}}.$$

Since $\phi(e) = 1$ we have $\phi J \phi^{-1} = K$. Moreover, $\phi \tau'_x \phi^{-1} = \tau_{\phi(x)}$: indeed, $\tau_{\phi(x)}$ fixes \mathbb{H} pointwise, and so does $\phi \tau'_x \phi^{-1}$ since ϕ^{-1} maps \mathbb{H} to Q . Moreover,

$$\phi \tau'_x \phi^{-1}(z) = \phi \tau'_x(w) = \phi(wx) = \phi(w)\phi(x) = z\phi(x) = \tau_{\phi(x)}(z).$$

Likewise, $\phi \tau'_y \phi^{-1} = \tau_{\phi(y)}$, whence $A \simeq \mathbb{O}_{K^j \tau_a, K^i \tau_b}$ for some $(a, b) \in \mathbb{S}^3 \times \mathbb{S}^3$. Moreover, Proposition 19 in [19] states that $C_{J^j \tau'_x, J^i \tau'_y} \in \mathcal{D}_{1,3,4}$ if and only if, firstly, $(x, y) \neq (1, 1)$ and, secondly, $(i, j) = (1, 1)$ implies either $x^2 + x + e \neq 0$ or $y \neq x^2$ in C . This proves that $(a, b) \in (\mathbb{S}^3 \times \mathbb{S}^3)_{ij}$ in all cases. The converse holds by applying Proposition 19 from [19] with $C = \mathbb{O}$ and $(x, y) = (a, b)$. \square

In our construction of a description, we will let the group $\mathbb{S}^3 \rtimes SO_3$ from Lemma 3.6 act on the set $(\mathbb{S}^3 \times \mathbb{S}^3)_{ij}$ for each $(i, j) \in \mathbb{Z}_2^2$. The details are as follows.

Proposition 3.9. *Let $(i, j) \in \mathbb{Z}_2^2$. The group $\mathbb{S}^3 \rtimes SO_3$ acts on $(\mathbb{S}^3 \times \mathbb{S}^3)_{ij}$ by*

$$(3.4) \quad (p, \kappa_q) \cdot (a, b) = (pqa\overline{pq}, pqb\overline{pq}).$$

The map

$$\mathcal{G}^{ij} : \mathbb{S}^3 \rtimes SO_3 \rightarrow \mathcal{D}_{1,3,4}^{ij}$$

acting on objects by

$$(a, b) \mapsto \mathbb{O}_{K^j \tau_a, K^i \tau_b}$$

and on morphisms by $(p, \kappa_q) \mapsto \Delta((p, \kappa_q))$, is an equivalence of categories.

The isomorphism $\Delta : \mathbb{S}^3 \rtimes SO_3 \rightarrow G_2^{\mathbb{H}}$ was defined in Lemma 3.6 by $(p, \kappa_q) \mapsto \tau_{\widehat{p}} \widehat{\kappa_q}$.

Proof. By definition of the semi-direct product, (3.4) defines an action α of $\mathbb{S}^3 \rtimes SO_3$ on $\mathbb{S}^3 \times \mathbb{S}^3$. The point $(1, 1)$ is a fixed point of α , while for each $p, q \in \mathbb{S}^3$,

$$(p, \kappa_q) \cdot (a, a^2) = (pqa\overline{pq}, pqa^2\overline{pq}) = (pqa\overline{pq}, (pqa\overline{pq})^2),$$

with

$$(pqa\overline{pq})^2 + pqa\overline{pq} + 1 = pq(a^2 + a + 1)\overline{pq}.$$

Thus α induces an action of $\mathbb{S}^3 \rtimes SO_3$ on $(\mathbb{S}^3 \times \mathbb{S}^3)_{ij}$ for each $(i, j) \in \mathbb{Z}_2^2$.

Fix now $(i, j) \in \mathbb{Z}_2^2$. By Lemma 3.8, \mathcal{G}^{ij} is well-defined on objects. Next we show well-definedness on morphisms. First, identities are mapped to identities. Using (3.3) and the fact that $\widehat{\kappa}_q, \tau_p \in G_2$ for each $p, q \in \mathbb{S}^3$, one obtains, for each $w \in \mathbb{S}^3$, that

$$(3.5) \quad \tau_{\overline{p}} \widehat{\kappa}_q \tau_w (\tau_{\overline{p}} \widehat{\kappa}_q)^{-1} = \tau_{pqw\overline{p}q}.$$

Thus if $(p, \kappa_q) \cdot (a, b) = (c, d)$, then $\phi = \tau_{\overline{p}} \widehat{\kappa}_q \in G_2$ satisfies

$$\phi K^j \tau_a \phi^{-1} = K^j \tau_c \quad \text{and} \quad \phi K^i \tau_b \phi^{-1} = K^i \tau_d, \quad .$$

since ϕ commutes with K . By Remark 2.5, this proves that \mathcal{G}^{ij} maps morphisms to morphisms. Functoriality follows from the fact that Δ is a group homomorphism, and faithfulness from the fact that Δ is injective.

To show that \mathcal{G}^{ij} is full, let

$$\phi : A = \mathbb{O}_{K^j \tau_a, K^i \tau_b} \rightarrow B = \mathbb{O}_{K^j \tau_c, K^i \tau_d}$$

with $(a, b), (c, d) \in (\mathbb{S}^3 \times \mathbb{S}^3)_{ij}$. From [19] we know that $A_0 = B_0 = \langle 1 \rangle$, and that \mathbb{H} is a submodule of both A and B . From Remark 1.11 it therefore follows that $\phi(1) = \pm 1$. By Remark 2.5 we have $\phi \in G_2$ and $(\phi \tau_a \phi^{-1}, \phi \tau_b \phi^{-1}) = (\tau_c, \tau_d)$. Remark 1.11 also implies that $\phi(\mathbb{H}) = \mathbb{H}$. Lemma 3.6 then gives that $\phi = \Delta(p, \kappa_q)$ for some $p, q \in \mathbb{S}^3$, whence by (3.5),

$$\tau_{pqa\overline{p}q} = \tau_c \quad \text{and} \quad \tau_{pqb\overline{p}q} = \tau_d,$$

which by Lemma 3.5 implies that $c = pqa\overline{p}q$ and $d = pqb\overline{p}q$. Thus $\phi = \mathcal{G}^{ij}(p, \kappa_q)$ with $(p, \kappa_q) \cdot (a, b) = (c, d)$, and fullness is proved. Finally, \mathcal{G}^{ij} is dense by Lemma 3.8, and the proof is complete. \square

Remark 3.10. Let $(i, j) \in \mathbb{Z}_2^2$. Since $\mathcal{D}_{1,3,4}$ is the coproduct of the two subcategories $\mathcal{D}_{1,3,4}^s$ and $\mathcal{D}_{1,3,4}^a$, it may be called for to specify for which $(a, b) \in (\mathbb{S}^3 \times \mathbb{S}^3)_{ij}$ the image $\mathcal{G}^{ij}(a, b)$ belongs to either one. This information can be deduced immediately from the above cited Proposition 19 in [19]. Namely, $\mathcal{G}^{ij}(a, b) \in \mathcal{D}_{1,3,4}^s$ if $\{a, b\} \subseteq \{\pm 1\}$, and $\mathcal{G}^{ij}(a, b) \in \mathcal{D}_{1,3,4}^a$ otherwise.

3.4. The Subcategories $\mathcal{D}_{1,1,2,4}$ and $\mathcal{D}_{1,1,1,1,4}$. We define, for each $a, b \in \mathbb{H} \subset \mathbb{O}$ and each $k \in \mathbb{Z}_2$, the map $T_{a,b}^{(k)} : \mathbb{O} \rightarrow \mathbb{O}$ by

$$T_{a,b}^{(k)}(x) = \begin{cases} aK^k(x)b & \text{if } x \in \mathbb{H}, \\ x & \text{if } x \in \mathbb{H}^\perp. \end{cases}$$

For each $(i, j) \in \mathbb{Z}_2^2$, the next lemma gives a set exhausting $\mathcal{D}_{1,1,2,4}^{ij} \amalg \mathcal{D}_{1,1,1,1,4}^{ij}$.

Lemma 3.11. *Let $(i, j) \in \mathbb{Z}_2^2$ and $A \in \mathcal{D}_{1,1,2,4}^{ij} \amalg \mathcal{D}_{1,1,1,1,4}^{ij}$. Then A is isomorphic to $\mathbb{O}_{T_{a_1,b_1}^{(j)}, T_{a_2,b_2}^{(i)}}$ for some $(a_1, b_1, a_2, b_2) \in S_{ij}$. Conversely,*

$$\mathbb{O}_{T_{a_1,b_1}^{(j)}, T_{a_2,b_2}^{(i)}} \in \mathcal{D}_{1,1,2,4}^{ij} \amalg \mathcal{D}_{1,1,1,1,4}^{ij}$$

for each $(i, j) \in \mathbb{Z}_2^2$ and $(a_1, b_1, a_2, b_2) \in S_{ij}$.

For each $(i, j) \in \mathbb{Z}_2^2$, S_{ij} is defined as the set of all $(a_1, b_1, a_2, b_2) \in (\mathbb{S}^3)^4 \setminus \{\pm 1\}^4$ which do *not* satisfy

$$\dim(\Im(a_k), \Im(b_k) | k \in \{1, 2\}) = 1 \wedge a_1 = (-1)^j b_1 \wedge a_2 = (-1)^i b_2,$$

where $\Im : \mathbb{O} \rightarrow \mathbb{O}$ is the orthogonal projection onto 1^\perp .

The proof is similar in spirit to that of Lemma 3.8; the details are as follows.

Proof. Let $A \in \mathcal{D}_{1,1,2,4}^{ij} \amalg \mathcal{D}_{1,1,1,1,4}^{ij}$. By [19] (Proposition 40), there exists $C \in \mathcal{A}_8$ with unity e , a fixed quaternion subalgebra Q , and elements $x_1, x_2, y_1, y_2 \in Q$ with $\|x_i y_i\| = 1$ for each $i \in \{1, 2\}$, such that

$$A \simeq C_{T_{x_1, y_1}^{(l)}, T_{x_2, y_2}^{(k)}}$$

for some $(k, l) \in \mathbb{Z}_2^2$. The map $T_{x, y}^{(m)} : C \rightarrow C$ is defined for each $m \in \mathbb{Z}_2$ and each $x, y \in \mathbb{S}(Q)$ by

$$T_{x, y}^{(m)}(w) = \begin{cases} xJ^m(w)y & \text{if } w \in Q, \\ w & \text{if } w \in Q^\perp, \end{cases}$$

where J is the standard involution on C . Proposition 1.3 then gives $(k, l) = (i, j)$. As before, Lemma 2.1 guarantees the existence of an isomorphism

$$\phi : C \rightarrow \mathbb{O}$$

mapping e to 1 and Q to \mathbb{H} . Thence by Lemma 2.2 we have the isomorphism

$$\phi : C_{T_{x_1, y_1}^{(j)}, T_{x_2, y_2}^{(i)}} \rightarrow \mathbb{O}_{\phi T_{x_1, y_1}^{(j)} \phi^{-1}, \phi T_{x_2, y_2}^{(i)} \phi^{-1}}.$$

Note that upon rescaling x and y we may assume that $\|x_i\| = \|y_i\| = 1, i \in \{1, 2\}$. Next we prove that $\phi T_{x, y}^{(m)} \phi^{-1} = T_{\phi(x), \phi(y)}^{(m)}$ for each $m \in \mathbb{Z}_2$ and $x, y \in Q$ with $\|x\| = \|y\| = 1$. Since ϕ maps Q to \mathbb{H} , and therefore Q^\perp to \mathbb{H}^\perp , the left hand side fixes \mathbb{H}^\perp pointwise, and as does the right hand side. Given any $w \in \mathbb{H}$ we obtain, using the fact that $\phi : C \rightarrow \mathbb{O}$ is an isomorphism, that

$$\phi T_{x, y}^{(m)} \phi^{-1}(w) = \phi(xJ^m \phi^{-1}(w)y) = \phi(x)K^m(w)\phi(y) = T_{\phi(x), \phi(y)}^{(m)}(w).$$

Thus $A \simeq \mathbb{O}_{T_{a_1, b_1}^{(j)}, T_{a_2, b_2}^{(i)}}$ for some $(a_1, b_1, a_2, b_2) \in (\mathbb{S}^3)^4$. Now Proposition 28 in [19] implies that $\mathbb{O}_{T_{a_1, b_1}^{(j)}, T_{a_2, b_2}^{(i)}} \in \mathcal{D}_{1,1,2,4}^{ij} \amalg \mathcal{D}_{1,1,1,1,4}^{ij}$ precisely when $(a_1, b_1, a_2, b_2) \in S_{ij}$, which completes the argument. The converse holds by applying Proposition 28 from [19] with $C = \mathbb{O}$ and $(x_1, y_1, x_2, y_2) = (a_1, b_1, a_2, b_2)$. \square

To obtain a description, we will again make use of the group $\mathbb{S}^3 \rtimes SO_3$, acting on a set that we will now construct. Consider thus the normal subgroup $\{\pm(1, 1)\}$ of $\mathbb{S}^3 \times \mathbb{S}^3$, and denote the corresponding quotient group by $[\mathbb{S}^3 \times \mathbb{S}^3]$, and its elements by $[a, b]$ with $a, b \in \mathbb{S}^3$. We obtain a surjective map

$$(\mathbb{S}^3)^4 \rightarrow [\mathbb{S}^3 \times \mathbb{S}^3] \times [\mathbb{S}^3 \times \mathbb{S}^3], (a_1, b_1, a_2, b_2) \mapsto ([a_1, b_1], [a_2, b_2]),$$

and denote, for each $(i, j) \in \mathbb{Z}_2^2$, the image of S_{ij} under this map by $[S_{ij}]$.

Proposition 3.12. *Let $i, j \in \mathbb{Z}_2$. The group $\mathbb{S}^3 \rtimes SO_3$ acts on $[S_{ij}]$ by*

$$(3.6) \quad (p, \kappa_q) \cdot ([a_1, b_1], [a_2, b_2]) = ([\kappa_q(a_1), \kappa_q(b_1)], [\kappa_q(a_2), \kappa_q(b_2)]).$$

The map

$$\mathcal{I}^{ij} : \mathbb{S}^3 \rtimes SO_3[S_{ij}] \rightarrow \mathcal{D}_{1,1,2,4}^{ij} \amalg \mathcal{D}_{1,1,1,1,4}^{ij}$$

acting on objects by

$$([a_1, b_1], [a_2, b_2]) \mapsto \mathbb{O}_{T_{a_1, b_1}^{(j)}, T_{a_2, b_2}^{(i)}}$$

and on morphisms by $(p, \kappa_q) \mapsto \Delta((p, \kappa_q))$, is an equivalence of categories.

The map Δ was defined in Lemma 3.6.

Proof. Since κ_q is an automorphism of \mathbb{H} , (3.6) defines an action of $\mathbb{S}^3 \rtimes SO_3$ on $[\mathbb{S}^3 \times \mathbb{S}^3] \times [\mathbb{S}^3 \times \mathbb{S}^3]$, mapping $[S_{ij}]$ to itself for each $(i, j) \in \mathbb{Z}_2^2$.

Let now $(i, j) \in \mathbb{Z}_2^2$. The map \mathcal{I}^{ij} is well-defined on objects by Lemma 3.11 and the fact that for each $k \in \mathbb{Z}_2$ and each $a, b \in \mathbb{S}^3$,

$$T_{a,b}^{(k)} = T_{-a,-b}^{(k)}.$$

To show well-definedness on morphisms we first note that identities are mapped to identities. Moreover, for each $a, b, p, q \in \mathbb{S}(\mathbb{H})$ and each $k \in \mathbb{Z}_2$,

$$(3.7) \quad \tau_{\widehat{p}\widehat{\kappa}_q} T_{a,b}^{(k)} (\tau_{\widehat{p}\widehat{\kappa}_q})^{-1} = T_{qa\bar{q},qb\bar{q}}^{(k)}.$$

Thus if

$$(p, \kappa_q) \cdot ([a_1, b_1], [a_2, b_2]) = ([c_1, d_1], [c_2, d_2])$$

then for $\phi = \tau_{\widehat{p}\widehat{\kappa}_q} \in G_2$ we have

$$\phi T_{a_1,b_1}^{(j)} \phi^{-1} = T_{c_1,d_1}^{(j)} \quad \text{and} \quad \phi T_{a_2,b_2}^{(i)} \phi^{-1} = T_{c_2,d_2}^{(i)}.$$

Since $T_{a,b}^{(k)}$ acts as identity on \mathbb{H}^\perp for each $k \in \mathbb{Z}_2$ and $a, b \in \mathbb{S}^3$, and since ϕ maps \mathbb{H}^\perp to itself, we may apply Lemma 2.4 to deduce that ϕ is indeed a morphism from $\mathbb{O}_{T_{a_1,b_1}^{(j)}, T_{a_2,b_2}^{(i)}}$ to $\mathbb{O}_{T_{c_1,d_1}^{(j)}, T_{c_2,d_2}^{(i)}}$. Functoriality and faithfulness follow from Δ being a group monomorphism, and denseness from Lemma 3.11. To see that the functor is full, let $A = \mathbb{O}_{T_{a_1,b_1}^{(j)}, T_{a_2,b_2}^{(i)}}$ and $B = \mathbb{O}_{T_{c_1,d_1}^{(j)}, T_{c_2,d_2}^{(i)}}$ for some (a_1, b_1, a_2, b_2) and (c_1, d_1, c_2, d_2) in S_{ij} . From [19] it follows that \mathbb{H}^\perp is an irreducible submodule of dimension four of A as well as of B . Thus by Remark 1.11, any isomorphism $\phi : A \rightarrow B$ maps \mathbb{H}^\perp to itself. We may then apply Lemma 2.4 to conclude that $\phi \in G_2$ and that

$$\phi T_{a_1,b_1}^{(j)} \phi^{-1} = T_{c_1,d_1}^{(j)} \quad \text{and} \quad \phi T_{a_2,b_2}^{(i)} \phi^{-1} = T_{c_2,d_2}^{(i)}.$$

Lemma 3.6 further implies that $\phi = \Delta((p, \kappa_q))$ for some $p, q \in \mathbb{S}^3$, and by (3.7) we have

$$(3.8) \quad T_{qa_1\bar{q},qb_1\bar{q}}^{(j)} = T_{c_1,d_1}^{(j)} \quad \text{and} \quad T_{qa_2\bar{q},qb_2\bar{q}}^{(i)} = T_{c_2,d_2}^{(i)}.$$

Now the map $(a, b) \mapsto T_{a,b}^{(k)}$ is a 2-1-map from $\mathbb{S}^3 \times \mathbb{S}^3$ to the set of all isometries of \mathbb{H} with determinant k , the preimage of $T_{a,b}^{(k)}$ being $\{\pm(a, b)\}$. Thus (3.8) implies that

$$([qa_1\bar{q}, qb_1\bar{q}], [qa_2\bar{q}, qb_2\bar{q}]) = ([c_1, d_1], [c_2, d_2]).$$

Altogether, for each $\phi : A \rightarrow B$ there exist $p, q \in \mathbb{S}^3$ with

$$(p, \kappa_q) \cdot ([a_1, b_1], [a_2, b_2]) = ([c_1, d_1], [c_2, d_2]),$$

such that $\phi = \mathcal{I}^{ij}(p, \kappa_q)$. This proves fullness, whereby the proof is complete. \square

Remark 3.13. Similarly to the previous case we have, for each $(i, j) \in \mathbb{Z}_2^2$, a description of the coproduct of two subcategories without it being obvious which objects belong to one or the other. From Proposition 28 in [19] we can however read off that $\mathcal{I}^{ij}([a_1, b_1], [a_2, b_2])$ is in $\mathcal{D}_{1,1,2,4}$ if $\dim[\Im(a_k), \Im(b_k)]k \in \{1, 2\} = 1$, and in $\mathcal{D}_{1,1,1,1,4}$ otherwise. Descriptions of $\mathcal{D}_{1,1,2,4}^{ij}$ and $\mathcal{D}_{1,1,1,1,4}^{ij}$ for each $(i, j) \in \mathbb{Z}_2^2$ can thus be obtained by restricting the functor \mathcal{I}^{ij} accordingly.

4. QUASI-DESCRIPTIONS

A description of a groupoid \mathcal{C} takes into account all isomorphisms in \mathcal{C} . As we saw above, for subcategories of \mathcal{D} the descriptions involve a large amount of information. In this section we introduce the notion of a quasi-description, and apply it to the groupoid \mathcal{D} . The idea is that for classification purposes, it is not necessary to consider all isomorphisms between the objects of a groupoid, as it suffices to detect whether or not there exists an isomorphism between two objects. In the case of the groupoid \mathcal{D} , this gives, as we will see, a simpler and more unified approach, from which we will be able to obtain a classification relatively easily.

4.1. Preliminaries. We begin by defining the general set-up.

Definition 4.1. A functor $\mathcal{F} : \mathcal{B} \rightarrow \mathcal{C}$ between two categories \mathcal{B} and \mathcal{C} is said to *detect non-isomorphic objects* if for any $B, B' \in \mathcal{B}$,

$$\mathcal{F}(B) \simeq \mathcal{F}(B') \implies B \simeq B'.$$

Note that the inverse implication holds for all functors. If \mathcal{B} and \mathcal{C} are groupoids, then $\mathcal{F} : \mathcal{B} \rightarrow \mathcal{C}$ detects non-isomorphic objects if and only if each non-empty morphism class in \mathcal{C} contains at least one morphism that is in the image of \mathcal{F} .

Example 4.2. Any full functor between groupoids detects non-isomorphic objects.

Proposition 4.3. Let $\mathcal{F} : \mathcal{B} \rightarrow \mathcal{C}$ be a dense functor between two categories \mathcal{B} and \mathcal{C} which detects non-isomorphic objects. If an object class $\mathcal{S} \subseteq \mathcal{B}$ classifies \mathcal{B} , then $\mathcal{F}(\mathcal{S})$ classifies \mathcal{C} .

Here we say that a subclass \mathcal{S} of a category \mathcal{B} *classifies \mathcal{B}* or *is a classification of \mathcal{B}* if \mathcal{S} is the object class of a skeleton of \mathcal{B} , i.e. if \mathcal{S} is dense in \mathcal{B} and different objects in \mathcal{S} are non-isomorphic.

Proof. If $C \in \mathcal{C}$, then $C \simeq \mathcal{F}(B)$ for some $B \in \mathcal{B}$ by denseness of \mathcal{F} , and $B \simeq S$ for some $S \in \mathcal{S}$ as \mathcal{S} is a classification. Thus $C \simeq \mathcal{F}(B) \simeq \mathcal{F}(S)$, whence $\mathcal{F}(\mathcal{S})$ is dense in \mathcal{C} . Take now $C \neq C' \in \mathcal{F}(\mathcal{S})$. Then $C = \mathcal{F}(S)$ and $C' = \mathcal{F}(S')$ with $S \neq S' \in \mathcal{S}$. If $C \simeq C'$ in \mathcal{C} , then $S \simeq S'$ in \mathcal{B} as \mathcal{F} detects non-isomorphic objects. This contradicts \mathcal{S} being a classification of \mathcal{B} , whence necessarily $C \not\simeq C'$. \square

Remark 4.4. Given \mathcal{C} for which one can provide a category \mathcal{B} and a dense functor $\mathcal{F} : \mathcal{B} \rightarrow \mathcal{C}$ detecting non-isomorphic objects, the problem of finding a classification of \mathcal{C} carries over to that of \mathcal{B} . However, as the functor is in general neither faithful nor full, more precise information about the morphisms may be lost.

We now use this to generalize the concept of a description as follows.

Definition 4.5. A *quasi-description* of a category \mathcal{C} is a quadruple $(G, M, \alpha, \mathcal{F})$ where G is a group, M is a set, $\alpha : G \times M \rightarrow M$ a group action, and $\mathcal{F} : {}_G M \rightarrow \mathcal{C}$ a dense functor which detects non-isomorphic objects.

If \mathcal{C} is a groupoid, then every description of \mathcal{C} is a quasi-description. In general, the above proposition and remark imply that the problem of classifying \mathcal{C} is transferred, as in the case where there is a description, to the solution of the normal form problem for the action α .

Equipped with these tools, we return to composition algebras. Recall that for each $A \in \mathcal{D}$, we set

$$A_0 = \{a \in A \mid \forall \delta \in \text{Der}(A) : \delta(a) = 0\}.$$

We define the full subcategories \mathcal{L} and \mathcal{H} of \mathcal{D} with object classes

$$\mathcal{L} = \{A \in \mathcal{D} \mid \dim A_0 \leq 1\}$$

and

$$\mathcal{H} = \{A \in \mathcal{D} \mid \dim A_0 > 1\}.$$

Thus $\mathcal{D} = \mathcal{L} \amalg \mathcal{H}$. We will treat each of these subcategories separately. Before doing so, we define the group actions to be used in the quasi-descriptions. As before we identify \mathbb{S}^3 with $\mathbb{S}(\mathbb{H})$ and SO_3 with $\{\kappa_q \mid q \in \mathbb{S}^3\}$. Then SO_3 acts on $\mathbb{S}^3 \times \mathbb{S}^3$ by

$$\kappa_q \cdot (a, b) = (\kappa_q(a), \kappa_q(b)).$$

When there is no ambiguity, we will simply refer to this as *the action of SO_3 on $\mathbb{S}^3 \times \mathbb{S}^3$* . This defines the groupoid ${}_{SO_3}(\mathbb{S}^3 \times \mathbb{S}^3)$. Moreover, it induces the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3] = (\mathbb{S}^3 \times \mathbb{S}^3)/\{\pm(1, 1)\}$ given by

$$\kappa_q \cdot [a, b] = [\kappa_q(a), \kappa_q(b)].$$

This will in turn be referred to as *the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]$* , and defines the groupoid ${}_{SO_3}[\mathbb{S}^3 \times \mathbb{S}^3]$. Using this, the groupoid ${}_{SO_3}[\mathbb{S}^3 \times \mathbb{S}^3]^2$ is defined by

$$\kappa_q \cdot ([a_1, b_1], [a_2, b_2]) = ([\kappa_q(a_1), \kappa_q(b_1)], [\kappa_q(a_2), \kappa_q(b_2)]),$$

and the corresponding action will be referred to as *the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]^2$* .

4.2. Algebras with Low-Dimensional Trivial Submodule. From Proposition 1.9 we know that the category \mathcal{L} admits the coproduct decomposition

$$\mathcal{L} = \mathcal{D}_{1,7} \amalg \mathcal{D}_{\{8\}} \amalg \mathcal{D}_{1,3,4} \amalg \mathcal{D}_{3+5}.$$

In this section we will obtain a quasi-description of

$$\mathcal{L}_0 = \mathcal{D}_{1,7} \amalg \mathcal{D}_{\{8\}} \amalg \mathcal{D}_{1,3,4},$$

while $\mathcal{D}_{3,5}$ will be treated in Section 5. A quasi-description of each double sign component of \mathcal{L}_0 is obtained as follows.

Theorem 4.6. *Let $(i, j) \in \mathbb{Z}_2^2$. The map*

$$\mathcal{J}^{ij} : {}_{SO_3}(\mathbb{S}^3 \times \mathbb{S}^3) \rightarrow \mathcal{L}_0^{ij}$$

defined on objects by

$$(a, b) \mapsto \mathbb{O}_{K^j \tau_a, K^i \tau_b}$$

and on morphisms by $\kappa_q \mapsto \hat{\kappa}_q$, is a dense functor detecting non-isomorphic objects.

For each $q \in \mathbb{S}^3$, $\hat{\kappa}_q \in G_2$ was defined in Section 3 by $(u, v, z) \mapsto (\kappa_q(u), \kappa_q(v), z)$. The proof is a synthesis of the results and arguments of Sections 3.1-3.3.

Proof. The map \mathcal{J}^{ij} is well-defined on objects by [19], Proposition 19, and the fact that the double sign of $\mathbb{O}_{K^j \tau_a, K^i \tau_b}$ is (i, j) . Regarding morphisms, observe first that identities are mapped to identities. Next assume that $\kappa_q \cdot (a, b) = (c, d)$ for some $q, a, b, c, d \in \mathbb{S}^3$. From (3.5) with $p = 1$ we deduce that $\hat{\kappa}_q \in G_2$ satisfies

$$\hat{\kappa}_q K^j \tau_a \hat{\kappa}_q^{-1} = K^j \tau_c \quad \text{and} \quad \hat{\kappa}_q K^i \tau_b \hat{\kappa}_q^{-1} = K^i \tau_d.$$

Then by Remark 2.5, \mathcal{J}^{ij} maps morphisms to morphisms. Functoriality is clear since $\kappa_q \kappa_{q'} = \kappa_{qq'}$ and $\hat{\kappa}_q \hat{\kappa}_{q'} = \hat{\kappa}_{qq'}$ for any $q, q' \in \mathbb{S}^3$.

The intersection of the image of \mathcal{J}^{ij} with $\mathcal{D}_{1,7}^{ij}$ is dense in $\mathcal{D}_{1,7}^{ij}$ by Proposition 3.1, the intersection with $\mathcal{D}_{\{8\}}^{ij}$ is dense in $\mathcal{D}_{\{8\}}^{ij}$ by Example 3.3 and the comments

following it, and the intersection with $\mathcal{D}_{1,3,4}^{ij}$ is dense in $\mathcal{D}_{1,3,4}^{ij}$ by Proposition 3.9. Since \mathcal{L}_0^{ij} is the coproduct of these categories, this proves that \mathcal{J}^{ij} is dense.

It remains to be shown that \mathcal{J}^{ij} detects non-isomorphic objects. Assume thus that $(a, b), (c, d) \in \mathbb{S}^3 \times \mathbb{S}^3$ satisfy $A = \mathbb{O}_{K^j\tau_a, K^i\tau_b} \simeq \mathbb{O}_{K^j\tau_c, K^i\tau_d} = B$.

If $A, B \in \mathcal{D}_{1,7}^{ij}$, then by [19], Proposition 19, $a = b = c = d = 1$, whence obviously $(a, b) \simeq (c, d)$ in $SO_3(\mathbb{S}^3 \times \mathbb{S}^3)$.

If $(i, j) = (1, 1)$ and $A, B \in \mathcal{D}_{\{8\}}^{ij}$, then from Proposition 19 in [19] we have $a^2 + a = c^2 + c = -1$ and $b - a^2 = d - c^2 = 0$. Solving these equations one deduces that there exist $w, w' \in \mathbb{S}(1^\perp)$ such that

$$(a, b) = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}w, -\frac{1}{2} - \frac{\sqrt{3}}{2}w\right) \quad \text{and} \quad (c, d) = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}w', -\frac{1}{2} - \frac{\sqrt{3}}{2}w'\right).$$

Since SO_3 acts transitively on $\mathbb{S}(1^\perp) \subset \mathbb{H}$, there exists $q \in \mathbb{S}^3$ such that $\kappa_q(w) = w'$, whence $(a, b) \simeq (c, d)$.

Finally, if $A, B \in \mathcal{D}_{1,3,4}^{ij}$, then A and B lie in the image of the functor \mathcal{G}^{ij} of Proposition 3.9. Since that is an equivalence of categories, there exist $p, q \in \mathbb{S}^3$ such that

$$(c, d) = (pqap\overline{q}, pqbp\overline{q}),$$

whence $\kappa_{pq} \cdot (a, b) = (c, d)$, and $(a, b) \simeq (c, d)$ in $SO_3(\mathbb{S}^3 \times \mathbb{S}^3)$. \square

A transversal for the action of SO_3 on $\mathbb{S}^3 \times \mathbb{S}^3$ will be given in the final section.

4.3. Algebras with High-Dimensional Trivial Submodule. Proposition 1.9 implies that the category \mathcal{H} decomposes as

$$\mathcal{H} = \mathcal{D}_{1,1,6} \amalg \mathcal{D}_{1,1,2,4} \amalg \mathcal{D}_{1,1,1,1,4} \amalg \mathcal{D}_{1,1,3,3}.$$

We will give a quasi-description of

$$\mathcal{H}_0 = \mathcal{D}_{1,1,6} \amalg \mathcal{D}_{1,1,2,4} \amalg \mathcal{D}_{1,1,1,1,4},$$

while $\mathcal{D}_{1,1,3,3}$ will be treated in Section 5. The quasi-description is constructed in two steps, of which the following lemma is the first.

Lemma 4.7. *Let $(i, j) \in \mathbb{Z}_2^2$. The map*

$$\mathcal{K}^{ij} : SO_3[\mathbb{S}^3 \times \mathbb{S}^3]^2 \rightarrow \mathcal{D}^{ij}$$

defined on objects by

$$([a_1, b_1], [a_2, b_2]) \mapsto \mathbb{O}_{T_{a_1, b_1}^{(j)}, T_{a_2, b_2}^{(i)}}$$

and on morphisms by $\kappa_q \mapsto \widehat{\kappa}_q$, is a functor.

Proof. The map \mathcal{K}^{ij} is well-defined on objects by [19], Proposition 28, and by double sign considerations, and maps identities to identities. If

$$\kappa_q \cdot ([a_1, b_1], [a_2, b_2]) = ([c_1, d_1], [c_2, d_2]),$$

then applying (3.7) with $p = 1$ we see that $\widehat{\kappa}_q \in G_2$ satisfies

$$\widehat{\kappa}_q T_{a_1, b_1}^{(j)} \widehat{\kappa}_q^{-1} = T_{c_1, d_1}^{(j)} \quad \text{and} \quad \widehat{\kappa}_q T_{a_2, b_2}^{(i)} \widehat{\kappa}_q^{-1} = T_{c_2, d_2}^{(i)}.$$

Then Lemma 2.4 (with $U = \mathbb{H}^\perp$) implies that $\widehat{\kappa}_q : \mathbb{O}_{T_{a_1, b_1}^{(j)}, T_{a_2, b_2}^{(i)}} \rightarrow \mathbb{O}_{T_{c_1, d_1}^{(j)}, T_{c_2, d_2}^{(i)}}$, and thus \mathcal{K}^{ij} is well-defined on morphisms. Functoriality follows from the fact that $\kappa_q \kappa_{q'} = \kappa_{qq'}$ and $\widehat{\kappa}_q \widehat{\kappa}_{q'} = \widehat{\kappa}_{qq'}$ for any $q, q' \in \mathbb{S}^3$. \square

We wish to use these functors to obtain quasi-descriptions of \mathcal{H}_0 . However, for each $(i, j) \in \mathbb{Z}_2^2$, the image of \mathcal{K}^{ij} is not contained in \mathcal{H}_0 . This is made precise in the following remark.

Remark 4.8. By Proposition 28 in [19], $\mathbb{O}_{T_{a_1, b_1}^{(j)}, T_{a_2, b_2}^{(i)}} = \mathcal{K}^{ij}([a_1, b_1], [a_2, b_2])$ belongs to \mathcal{H}_0 if and only if

$$\{a_1, b_1, a_2, b_2\} \not\subseteq \{\pm 1\}.$$

The set of all $[a_1, b_1], [a_2, b_2]$ such that $\{a_1, b_1, a_2, b_2\} \subseteq \{\pm 1\}$ is pointwise fixed under the action of SO_3 . Call the complement of this set S . Then the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]^2$ induces an action on S , giving rise to the groupoid $_{SO_3}S$.

We will use the groupoid $_{SO_3}S$ in our quasi-descriptions as follows.

Theorem 4.9. *Let $(i, j) \in \mathbb{Z}_2^2$. The map*

$$\mathcal{K}_*^{ij} : _{SO_3}S \rightarrow \mathcal{H}_0^{ij}$$

defined on objects by

$$([a_1, b_1], [a_2, b_2]) \mapsto \mathbb{O}_{T_{a_1, b_1}^{(j)}, T_{a_2, b_2}^{(i)}}$$

and on morphisms by $\kappa_q \mapsto \widehat{\kappa}_q$, is a dense functor, detecting non-isomorphic objects.

The proof builds on the results of Sections 3.2 and 3.4.

Proof. The image of \mathcal{K}_*^{ij} is in \mathcal{H}_0^{ij} by Remark 4.8. Therefore \mathcal{K}_*^{ij} is a well-defined functor, being the restriction of the functor \mathcal{K}^{ij} from Lemma 4.7. We will now show that \mathcal{K}_*^{ij} is dense and detects non-isomorphic objects by considering its image in $\mathcal{D}_{1,1,6}$ and $\mathcal{D}_{1,1,2,4} \amalg \mathcal{D}_{1,1,1,4}$ separately, starting with the first.

If $A \in \mathcal{D}_{1,1,6}^{ij}$, then $A \simeq \mathbb{O}_{\lambda_{e_1}^{(j)}, \lambda_{e_2}^{(i)}}$ for some $e_1, e_2 \in \mathbb{C} \subset \mathbb{O}$, by Proposition 3.5. To prove that $\mathbb{O}_{\lambda_{e_1}^{(j)}, \lambda_{e_2}^{(i)}}$ is the image of some $([a_1, b_1], [a_2, b_2]) \in S$, note that there exist $\alpha_1, \alpha_2 \in [0, \pi)$ such that

$$e_1 = \cos(\pi j + 2\alpha_1) + u \sin(\pi j + 2\alpha_1), \quad e_2 = \cos(\pi i + 2\alpha_2) + u \sin(\pi i + 2\alpha_2).$$

Take such α_1 and α_2 and set, for each $m \in \{1, 2\}$, $a_m = \cos(\alpha_m) + u \sin(\alpha_m)$. Moreover set

$$b_1 = (-1)^j a_1 \quad \text{and} \quad b_2 = (-1)^i a_2.$$

By straightforward computations we see that $\lambda_{e_1}^{(j)} = T_{a_1, b_1}^{(j)}$ and $\lambda_{e_2}^{(i)} = T_{a_2, b_2}^{(i)}$. If $\{a_1, b_1, a_2, b_2\} \subset \langle 1 \rangle$, then $\alpha_1 = \alpha_2 = 0$, and $(e_1, e_2) = ((-1)^j, (-1)^i)$, which by Proposition 3.5 contradicts that $\mathbb{O}_{\lambda_{e_1}^{(j)}, \lambda_{e_2}^{(i)}} \in \mathcal{D}_{1,1,6}^{ij}$. Thus $\{a_1, b_1, a_2, b_2\} \not\subseteq \langle 1 \rangle$, whence $([a_1, b_1], [a_2, b_2]) \in S$ and $\mathbb{O}_{\lambda_{e_1}^{(j)}, \lambda_{e_2}^{(i)}} = \mathcal{K}_*^{ij}([a_1, b_1], [a_2, b_2])$.

Assume next that $\mathbb{O}_{T_{a_1, b_1}^{(j)}, T_{a_2, b_2}^{(i)}}, \mathbb{O}_{T_{c_1, d_1}^{(j)}, T_{c_2, d_2}^{(i)}} \in \mathcal{D}_{1,1,6}^{ij}$ are isomorphic. We will show that $([a_1, b_1], [a_2, b_2]) \simeq ([c_1, d_1], [c_2, d_2])$ in $_{SO_3}S$. By Proposition 28 in [19], $\mathbb{O}_{T_{a_1, b_1}^{(j)}, T_{a_2, b_2}^{(i)}} \in \mathcal{D}_{1,1,6}^{ij}$ implies that the subalgebra of \mathbb{O} generated by $\{a_1, b_1, a_2, b_2\}$ is two-dimensional, and that

$$b_1 = (-1)^j a_1 \quad \text{and} \quad b_2 = (-1)^i a_2.$$

Hence there exist $w \in \mathbb{S}(1^\perp)$ and $\alpha_1, \alpha_2 \in [0, 2\pi)$ such that for each $m \in \{1, 2\}$

$$a_m = \cos(\alpha_m) + w \sin(\alpha_m) \quad \text{and} \quad b_m = \cos(\pi k_m + \alpha_m) + w \sin(\pi k_m + \alpha_m),$$

where $k_1 = j$ and $k_2 = i$. Now, $w = \kappa_q(u)$ for some $q \in \mathbb{S}^3$. Therefore $([a_1, b_1], [a_2, b_2]) \simeq ([a'_1, b'_1], [a'_2, b'_2])$ in $SO_3 S$, where

$$a'_m = \cos(\alpha_m) + u \sin(\alpha_m) \quad \text{and} \quad b'_m = \cos(\pi k_m + \alpha_m) + u \sin(\pi k_m + \alpha_m),$$

Likewise, $([c_1, d_1], [c_2, d_2]) \simeq ([c'_1, d'_1], [c'_2, d'_2])$ in $SO_3 S$, with

$$c'_m = \cos(\beta_m) + u \sin(\beta_m) \quad \text{and} \quad d'_m = \cos(\pi k_m + \beta_m) + u \sin(\pi k_m + \beta_m),$$

for some $\beta_1, \beta_2 \in [0, 2\pi)$. For each $m \in \{1, 2\}$ we set

$$e_m = \cos(\pi k_m + 2\alpha_m) + u \sin(\pi k_m + 2\alpha_m)$$

and

$$f_m = \cos(\pi k_m + 2\beta_m) + u \sin(\pi k_m + 2\beta_m).$$

Then straightforward computations show that

$$\lambda_{e_1}^{(j)} = T_{a'_1, b'_1}^{(j)}, \quad \lambda_{e_2}^{(i)} = T_{a'_2, b'_2}^{(i)}, \quad \lambda_{f_1}^{(j)} = T_{c'_1, d'_1}^{(j)}, \quad \text{and} \quad \lambda_{f_2}^{(i)} = T_{c'_2, d'_2}^{(i)}.$$

Thus $\mathbb{O}_{T_{a_1, b_1}^{(j)}, T_{a_2, b_2}^{(i)}} \simeq \mathbb{O}_{T_{c_1, d_1}^{(j)}, T_{c_2, d_2}^{(i)}}$ implies that $\mathbb{O}_{\lambda_{e_1}^{(j)}, \lambda_{e_2}^{(i)}} \simeq \mathbb{O}_{\lambda_{f_1}^{(j)}, \lambda_{f_2}^{(i)}}$. Using Proposition 3.5 this gives

$$(e_1, e_2) = (f_1, f_2) \quad \text{or} \quad (e_1, e_2) = (\overline{f_1}, \overline{f_2}).$$

In both cases there exists $q \in \mathbb{S}^3$ such that $\kappa_q \cdot ([a'_1, b'_1], [a'_2, b'_2]) = ([c'_1, d'_1], [c'_2, d'_2])$. Altogether this implies that $([a_1, b_1], [a_2, b_2]) \simeq ([c_1, d_1], [c_2, d_2])$ in $SO_3 S$, which was what we desired.

Thus the part of the proof concerned with $\mathcal{D}_{1,1,6}$ is complete, and we turn to the category $\mathcal{D}_{1,1,2,4} \amalg \mathcal{D}_{1,1,1,4}$. By Lemma 3.11, the image $\mathcal{K}_*^{ij}([S_{ij}]) \subseteq \mathcal{K}_*^{ij}(S)$ is dense in this category. Assume next that

$$\mathbb{O}_{T_{a_1, b_1}^{(j)}, T_{a_2, b_2}^{(i)}}, \mathbb{O}_{T_{c_1, d_1}^{(j)}, T_{c_2, d_2}^{(i)}} \in \mathcal{K}_*^{ij}(S)$$

are isomorphic. By Proposition 28 in [19], these belong to $\mathcal{D}_{1,1,2,4}^{ij} \amalg \mathcal{D}_{1,1,1,4}^{ij}$ if and only if $([a_1, b_1], [a_2, b_2]), ([c_1, d_1], [c_2, d_2]) \in [S_{ij}]$. Then since the functor \mathcal{I}^{ij} of Proposition 3.12 is full, the algebras being isomorphic implies that there exists $q \in \mathbb{S}^3$ such that

$$([qa_1 \overline{q}, qb_1 \overline{q}], [qa_2 \overline{q}, qb_2 \overline{q}]) = ([c_1, d_1], [c_2, d_2]),$$

whence $([a_1, b_1], [a_2, b_2]) \simeq ([c_1, d_1], [c_2, d_2])$ in $SO_3 S$. This completes the part of the proof concerned with $\mathcal{D}_{1,1,2,4} \amalg \mathcal{D}_{1,1,1,4}$.

Since \mathcal{H}_0^{ij} is the coproduct of the two subcategories treated above, it follows that \mathcal{K}_*^{ij} is dense and detects non-isomorphic objects, and the proof is complete. \square

The category $SO_3 [\mathbb{S}^3 \times \mathbb{S}^3]^2$ will be classified in the final section.

5. ALGEBRAS WITH DECOMPOSITION $\{3, 5\}$ OR $\{1, 1, 3, 3\}$

We now come to those subcategories of \mathcal{D} which fall outside the above treatment via quasi-descriptions. One common property of the algebras in these subcategories is that their derivation algebras are of type \mathfrak{su}_2 . Moreover, as was constructively demonstrated in [19], they can all be expressed as isotopes of algebras having a derivation algebra of type \mathfrak{su}_3 . In this section we first consider $\mathcal{L} \setminus \mathcal{L}_0 = \mathcal{D}_{3,5}$, consisting of certain isotopes of Okubo algebras. We show that this category consists of three isomorphism classes, one in each double sign different from $(-, -)$. (Recall that Okubo algebras form the unique isomorphism class $\mathcal{D}_{\{8\}} = \mathcal{D}_{\{8\}}^{11}$.) We

thereafter consider $\mathcal{H} \setminus \mathcal{H}_0 = \mathcal{D}_{1,1,3,3}$, which is exhausted by isotopes of algebras in $\mathcal{D}_{1,1,6}$. We show that these are classified by twelve two-parameter families of algebras, three in each double sign. (Recall here that $\mathcal{D}_{1,1,6}$ itself was classified by four two-parameter families, one in each double sign.)

5.1. The Category $\mathcal{D}_{3,5}$. The isotopes in $\mathcal{D}_{3,5}$ were constructed in [19] as follows. Let $P \in \mathcal{A}_8$ be an Okubo algebra, and $e \in P$ a non-zero idempotent. Then $C = P_{(R_e^P)^{-1}, (L_e^P)^{-1}}$ is unital with unity e , and we denote its multiplication by juxtaposition and its standard involution by J . Since P is an Okubo algebra, it is known from [13] that $(L_e^P)^{-1} = R_e^P = J\tilde{\tau}$ for some automorphism $\tilde{\tau}$ of C of order three, such that the fixed point set of $\tilde{\tau}$ is a unital four-dimensional subalgebra Q of C . Choose now $x \in Q^\perp$ of norm one. Then there is a unique $y \in Q$ such that $\tilde{\tau}(x) = xy$, and we choose $a \in Q$ of norm one orthogonal to e and y . For any such choice of x and a we call the span of $\{a, x, ax\}$ a *special subspace* of P .

Let now $P \in \mathcal{A}_8$ be an Okubo algebra and S a special subspace of P . We may then construct, for each $(i, j) \in \mathbb{Z}_2$, the isotope $P_{\sigma_S^{1-j}, \sigma_S^{1-i}}$, where σ_S acts as -1 on S and as identity on S^\perp . (If $(i, j) = (1, 1)$ this is just P itself.) It was proven in [19] that $P_{\sigma_S^j, \sigma_S^i} \in \mathcal{D}_{3,5}$ for each $(i, j) \neq (1, 1)$ and, conversely, that each $A \in \mathcal{D}_{3,5}$ is isomorphic to $P_{\sigma_S^{1-j}, \sigma_S^{1-i}}$ for an Okubo algebra $P \in \mathcal{A}_8$, a special subspace S of P , and a pair $(i, j) \neq (1, 1)$. Note that the double sign of $P_{\sigma_S^{1-j}, \sigma_S^{1-i}}$ is $((-1)^i, (-1)^j)$, since P has double sign $(-, -)$.

Example 5.1. The division Okubo algebra P^{11} was defined in Example 3.3 by

$$P^{11} = \mathbb{O}_{K\tau, K\tau^{-1}},$$

where $\tau = \tau_{(\sqrt{3}u-1)/2}$. One can check that $W = \langle v, z, vz \rangle$ is a special subspace of P^{11} . Thus for each $(i, j) \in \mathbb{Z}_2^2$ different from $(1, 1)$, the isotope

$$P^{ij} = \mathbb{O}_{K\tau\sigma_W^{1-j}, K\tau^{-1}\sigma_W^{1-i}}$$

belongs to $\mathcal{D}_{3,5}$.

The next result shows that the choices of P and S above are immaterial, and classifies $\mathcal{D}_{3,5}$. Recall that we have fixed a Cayley triple (u, v, z) in \mathbb{O} .

Theorem 5.2. *Let $(i, j) \in \mathbb{Z}_2^2$ and let P^{ij} be defined as in Example 5.1. The category $\mathcal{D}_{3,5}^{ij}$ is classified by $\{P^{ij}\}$, if $(i, j) \neq (1, 1)$, and $\mathcal{D}_{3,5}^{11} = \emptyset$.*

Proof. The emptiness of $\mathcal{D}_{3,5}^{11}$ and the fact that $P^{ij} \in \mathcal{D}_{3,5}^{ij}$ if $(i, j) \neq (1, 1)$ follow from the above discussion.

Let $(i, j) \neq (1, 1)$ and take $A \in \mathcal{D}_{3,5}^{ij}$. It remains to be shown that $A \simeq P^{ij}$. By the results from [19] recalled above, there exist P , e , $\tilde{\tau}$, Q , a and x as above, such that $S = \langle a, x, ax \rangle$ (where juxtaposition is multiplication in $C = P_{(R_e^P)^{-1}, (L_e^P)^{-1}}$) is a special subspace of P , and $A \simeq P_{\sigma_S^{1-j}, \sigma_S^{1-i}}$. Now

$$P = (P_{(R_e^P)^{-1}, (L_e^P)^{-1}})_{R_e^P, L_e^P} = C_{R_e^P, L_e^P} = CJ\tilde{\tau}, J\tilde{\tau}^{-1},$$

where J is the standard involution on the unital algebra C . By Lemma 2.1, there is an isomorphism $\psi : C \rightarrow \mathbb{O}$, mapping e to 1 and Q to \mathbb{H} , whence Lemma 2.2 supplies the isomorphism

$$\psi : P \rightarrow \mathbb{O}_{\psi J\tilde{\tau}\psi^{-1}, \psi J\tilde{\tau}^{-1}\psi^{-1}}.$$

Since $x \perp Q$ and has norm one, we have $\psi(x) \in \mathbb{S}(\mathbb{H}^\perp)$. Recall that $y \in Q$ is uniquely defined by $\tilde{\tau}(x) = xy$. Since $\tilde{\tau}$ has order three, we necessarily have $y^3 = e$ in C , and hence $\psi(y)^3 = 1$ in \mathbb{O} . Thus there exists $w \in \mathbb{S}(\mathbb{H})$ such that $w \perp 1$ and $\psi(y) = -1/2 + w\sqrt{3}/2$. Then $a \perp \langle e, y \rangle$ implies that $\psi(a) \in \mathbb{S}(\mathbb{H}) \cap \langle 1, w \rangle^\perp$, whence $(w, \psi(a), \psi(x))$ is a Cayley triple. Therefore there is a unique $\rho \in G_2$ mapping $(w, \psi(a), \psi(x))$ to (u, v, z) . Applying Lemma 2.2 to the composition $\phi = \rho\psi$ gives the isomorphism

$$\phi : P \rightarrow \mathbb{O}_{\phi J \tilde{\tau} \phi^{-1}, \phi J \tilde{\tau}^{-1} \phi^{-1}}.$$

Since ϕ is also an isomorphism from C to \mathbb{O} mapping Q to \mathbb{H} and y to $-1/2 + u\sqrt{3}/2$, it follows from the properties of $\tilde{\tau}$ that $\phi \tilde{\tau} \phi^{-1}$ is an automorphism of \mathbb{O} defined by fixing \mathbb{H} pointwise and mapping $z = \phi(x)$ to $\phi(xy) = z(\sqrt{3}u - 1)/2$. Thus $\phi \tilde{\tau} \phi^{-1} = \tau$. Moreover $\phi J \phi^{-1} = K$. Therefore

$$\mathbb{O}_{\psi J \tilde{\tau} \psi^{-1}, \psi J \tilde{\tau}^{-1} \psi^{-1}} = \mathbb{O}_{K\tau, K\tau^{-1}} = P^{11}.$$

A final application of Lemma 2.2 provides the isomorphism

$$\phi : P_{\sigma_S^{1-j}, \sigma_S^{1-i}} \rightarrow \mathbb{O}_{K\tau \phi \sigma_S^{1-j} \phi^{-1}, K\tau^{-1} \phi \sigma_S^{1-i} \phi^{-1}}.$$

Now $\phi \sigma_S \phi^{-1}$ acts as -1 on $\phi(S)$ and as identity on $\phi(S)^\perp$. Since $\phi(S)$ is the subspace $W = \langle v, z, vz \rangle$ from Example 5.1, this proves that $P_{\sigma_S^{1-j}, \sigma_S^{1-i}} \simeq P^{ij}$. Hence $A \simeq P^{ij}$, and the proof is complete. \square

In the above proof we in fact showed anew that each Okubo algebra $P \in \mathcal{A}_8$ is isomorphic to P^{11} . The reason for doing so, rather than merely quoting the fact, is that we needed to track the behaviour of the special subspace $S \subset P$ under the isomorphism.

Remark 5.3. Each P^{ij} is an isotope of $P^{11} = \mathcal{J}^{11}(t, t^2)$, where $t = (\sqrt{3}u - 1)/2$, and the functors \mathcal{J}^{ij} were defined in Section 4.2. However, it is not true, for $(i, j) \neq (1, 1)$, that $P^{ij} = \mathcal{J}^{ij}(t, t^2)$. If $(i, j) \neq (1, 1)$, then $\mathcal{J}^{ij}(t, t^2) = \mathcal{G}^{ij}(t, t^2)$, where \mathcal{G}^{ij} was defined in Proposition 3.9. Therefore, $\mathcal{J}^{ij}(t, t^2)$ belongs to $\mathcal{D}_{1,3,4}$. Indeed, (t, t^2) is in the domain $\mathbb{S}^3 \rtimes SO_3(\mathbb{S}^3 \times \mathbb{S}^3)_{ij}$ of \mathcal{G}^{ij} if and only if $(i, j) \neq (1, 1)$.

5.2. The Category $\mathcal{D}_{1,1,3,3}$. The below treatment of $\mathcal{D}_{1,1,3,3}$ is, as we will see, quite technical in nature. A dense subset of $\mathcal{D}_{1,1,3,3}$ was given in [19]. We begin by giving a refined version of it, after introducing some notation which we will use throughout this section.

Remark 5.4. Let $a \in \mathbb{S}(\mathbb{O})$. We write $L_a = L_a^\mathbb{O}$ and $R_a = R_a^\mathbb{O}$ for the maps $x \mapsto ax$ and $x \mapsto xa$, respectively. Since \mathbb{O} is alternative, $L_a R_a = R_a L_a$ and $L_a R_{\bar{a}} = R_{\bar{a}} L_a$, and we write B_a for the bimultiplication $L_a R_a$ and C_a for the conjugation $L_a R_{\bar{a}}$. (Note that $\bar{a} = a^{-1}$.) We further write σ_a for the reflection in the hyperplane a^\perp , i.e. σ_a is the linear map defined by $a \mapsto -a$ and $x \mapsto x$ for each $x \perp a$. As usual we have fixed a Cayley triple (u, v, z) in \mathbb{O} .

We moreover define, for each $\theta, \gamma \in \mathbb{R}$ and each $(k_1, k_2) \in \mathbb{Z}_2^2$, the map

$$G_{\theta, \gamma}^{k_1, k_2} = B_{\cos \theta + u \sin \theta} \sigma_u^{k_1} (\sigma_{uv} \sigma_{uz} \sigma_{vz \sin \gamma - (uv)z \cos \gamma})^{k_2} : \mathbb{O} \rightarrow \mathbb{O}.$$

Note that if $k_2 = 0$, then this map has eigenvalue 1 with eigenspace \mathbb{C}^\perp , and coincides with the map $\lambda_t^{(k_1)}$ from Section 3.2 with $t = \cos 2\theta + u \sin 2\theta$. If $k_2 = 1$, then 1 is an eigenvalue with eigenspace $W_\gamma = \langle v, z, vz \cos \gamma + (uv)z \sin \gamma \rangle$ and -1

is an eigenvalue with eigenspace uW_γ . In all cases, the matrix of $G_{\theta,\gamma}^{k_1,k_2}$ is 2×2 -block diagonal in the basis $(1, u, v, uv, z, uz, vz, (uv)z)$ of \mathbb{O} . Note also that since $B_a = B_{-a}$ we have $G_{\theta,\gamma}^{k_1,k_2} = G_{\theta+\pi n,\gamma}^{k_1,k_2}$ for all $n \in \mathbb{Z}$.

Lemma 5.5. *Let $A \in \mathcal{D}_{1,1,3,3}$. Then there exist $(i_1, i_2), (j_1, j_2) \in \mathbb{Z}_2^2$ with $i_2 = 1$ or $j_2 = 1$ and $\alpha, \beta, \gamma \in [0, \pi)$ such that $A \simeq \mathbb{O}_{f,g}$ with*

$$(5.1) \quad f = G_{\alpha,\gamma}^{j_1,j_2} \quad \text{and} \quad g = G_{\beta,\gamma}^{i_1,i_2}.$$

Moreover $((-1)^{i_1+i_2}, (-1)^{j_1+j_2})$ is the double sign of A .

Proof. Let $A \in \mathcal{D}_{1,1,3,3}$. By Proposition 39 in [19] there exists $C \in \mathcal{A}_8$ with unity e , an element $x \perp e$ and a three-dimensional subspace $S \subset \langle e, x \rangle^\perp$ with $S \perp xS$, such that $A \simeq C_{f',g'}$, where, f' and g' map each of $\langle e, x \rangle$, S and xS isometrically onto itself, act as identity on S , and satisfy

$$f'|_{xS} = (-1)^k \text{Id}|_{xS} \quad \text{and} \quad g'|_{xS} = (-1)^l \text{Id}|_{xS}$$

for some $(k, l) \in \mathbb{Z}_2^2$ with $k = 1$ or $l = 1$. Let (s_1, s_2, s_3) be an orthonormal basis of S . The condition that $S \perp xS$ implies in particular that $xs_1 \perp s_2$. If $\psi : C \rightarrow \mathbb{O}$ is an isomorphism, then the image of (x, s_1, s_2) is therefore a Cayley triple. Define $\rho \in G_2$ by $(\psi(x), \psi(s_1), \psi(s_2)) \mapsto (u, v, z)$, and set $\phi = \rho\psi$. Since ϕ is orthogonal, it maps s_3 into $\langle vz, (uv)z \rangle$, and for some $\gamma \in [0, \pi)$ we have $\langle \phi(s_3) \rangle = \langle vz \cos \gamma + (uv)z \sin \gamma \rangle$. Thus $\phi f \phi^{-1}$ maps each of $\langle 1, u \rangle$, W_γ and uW_γ isometrically onto itself, acting as the identity on W_γ , and as $(-1)^k$ times the identity on uW_γ . Setting $f = \phi f' \phi^{-1}$, it follows that f satisfies (5.1) for some suitable $\alpha \in [0, \pi)$ and $j_1 \in \mathbb{Z}_2$, and with $j_2 = k$. By analogous arguments, $g = \phi g' \phi^{-1}$ satisfies (5.1) for some $\beta \in [0, \pi)$ and $i_1 \in \mathbb{Z}_2$, and with $i_2 = l$. Lemma 2.2 then implies that $C_{f',g'} \simeq \mathbb{O}_{f,g}$. The statement about the double sign follows from Remark 1.4. \square

Observe that the converse is false, i.e. there exist f and g satisfying (5.1) with suitable parameters, such that $\mathbb{O}_{f,g} \notin \mathcal{D}_{1,1,3,3}$. We will return to this matter later.

We now have twelve 3-parameter families of algebras to consider. In order to simplify the classification problem, Alberto Elduque (personal communication, October 2013) showed that the parameter γ is superfluous, and further that each $A \in \mathcal{D}_{1,1,3,3}$ is isomorphic to $\mathbb{O}_{f,g}$ for some $f, g \in O_8$ fixing the element 1 and the subspace $\langle u \rangle$ and determined on $\langle 1, u \rangle^\perp$ by one parameter each. The precise content of his result is presented in the next proposition. Our proof however differs from his. For coherence with our approach we express the result in terms of the conjugation map defined in Remark 5.4. As in the previous section we set $W = \langle v, z, vz \rangle$ and write σ_W for $\sigma_{uv}\sigma_{uz}\sigma_{(uv)z}$. We define, for each $\theta \in \mathbb{R}$ and each $(k_1, k_2) \in \mathbb{Z}_2^2$, the map

$$F_\theta^{k_1,k_2} = C_{\cos \theta + u \sin \theta} \sigma_u^{k_1} \sigma_W^{k_2} : \mathbb{O} \rightarrow \mathbb{O}.$$

In the basis $(1, u, v, uv, z, uz, vz, (uv)z)$, this map has the block-diagonal matrix

$$\begin{pmatrix} 1 & & & & & & & \\ & k_1 & & & & & & \\ & & \widehat{2\theta}_{k_2} & & & & & \\ & & & \widehat{2\theta}_{k_2} & & & & \\ & & & & \widehat{-2\theta}_{k_2} & & & \\ & & & & & \widehat{-2\theta}_{k_2} & & \end{pmatrix},$$

where for each $\zeta \in \mathbb{R}$ and each $k \in \mathbb{Z}_2$,

$$\widehat{\zeta}_k = \begin{pmatrix} \cos \zeta & -(-1)^k \sin \zeta \\ \sin \zeta & (-1)^k \cos \zeta \end{pmatrix}.$$

Proposition 5.6. *Let $A \in \mathcal{D}_{1,1,3,3}$. Then $A \simeq \mathbb{O}_{F_\xi^{j_1, j_2}, F_\eta^{i_1, i_2}}$ for some $\xi, \eta \in \mathbb{R}$ and $(i_1, i_2), (j_1, j_2) \in \mathbb{Z}_2^2$ with $i_2 = 1$ or $j_2 = 1$.*

Proof. By Lemma 5.5 there exist $\alpha, \beta, \gamma \in [0, \pi)$ such that $A \simeq \mathbb{O}_{G_{\alpha, \gamma}^{j_1, j_2}, G_{\beta, \gamma}^{i_1, i_2}}$. Let $c = \cos(\gamma/3) - u \sin(\gamma/3)$, and define $\rho \in G_2$ by $(u, v, z) \mapsto (u, cv, cz)$, which is clearly a Cayley triple. Let $t = \cos \theta + u \sin \theta$ and $s = \cos \zeta + u \sin \zeta$ for $\theta, \zeta \in \mathbb{R}$ to be chosen later, and set $\phi = L_t R_s \rho$. Then

$$(\phi_1, \phi_2) = (B_t R_{\bar{s}} \rho, L_{\bar{t}} B_s \rho)$$

is a pair of triality components for ϕ , and $A \simeq \mathbb{O}_{\phi_1 f \phi^{-1}, \phi_2 g \phi^{-1}}$ by Proposition 2.3. We now choose θ and ζ so that

$$(5.2) \quad (3\theta/2, 3\zeta/2) = \begin{cases} (\alpha + 2\beta, 2\alpha + \beta) & \text{if } (i_1, j_1) = (0, 0), \\ (-\alpha, -2\alpha - 3\beta) & \text{if } (i_1, j_1) = (0, 1), \\ (-3\alpha - 2\beta, -\beta) & \text{if } (i_1, j_1) = (1, 0), \\ (-\alpha, -\beta) & \text{if } (i_1, j_1) = (1, 1). \end{cases}$$

Then by direct computations one verifies that

$$\phi_1 f \phi^{-1} = F_\xi^{j_1, j_2} \quad \text{and} \quad \phi_2 g \phi^{-1} = F_\eta^{i_1, i_2},$$

where $\xi, \eta \in \mathbb{R}$ are given by

$$(5.3) \quad (2\xi, 2\eta) = \begin{cases} (\theta - 2\gamma/3, \zeta - 2\theta) & \text{if } (i_2, j_2) = (0, 1), \\ (2\zeta - \theta, -\zeta - 2\gamma/3) & \text{if } (i_2, j_2) = (1, 0), \\ (\theta - 2\gamma/3, -\zeta - 2\gamma/3) & \text{if } (i_2, j_2) = (1, 1). \end{cases}$$

The proof is complete. \square

Remark 5.7. Let $(i_1, i_2), (j_1, j_2) \in \mathbb{Z}_2^2$ with $i_2 = 1$ or $j_2 = 1$. For any $\xi, \eta \in \mathbb{R}$ and any $\gamma \in [0, \pi)$, we can solve for α and β in (5.2) and (5.3). In particular it follows that for any $\xi, \eta \in \mathbb{R}$ there exist $\alpha, \beta \in [0, \pi)$ such that $\mathbb{O}_{F_\xi^{j_1, j_2}, F_\eta^{i_1, i_2}}$ is isomorphic to $\mathbb{O}_{G_{\alpha, 0}^{j_1, j_2}, G_{\beta, 0}^{i_1, i_2}}$. In other words, the parameter γ is superfluous.

The above proposition and remark together show that for each $A \in \mathcal{D}_{1,1,3,3}$ there exist $\alpha, \beta \in [0, \pi)$ and $(i_1, i_2), (j_1, j_2) \in \mathbb{Z}_2^2$ with $i_2 = 1$ or $j_2 = 1$ such that $A \simeq \mathbb{O}_{G_{\alpha, 0}^{j_1, j_2}, G_{\beta, 0}^{i_1, i_2}}$. The following lemma provides a converse to this statement.

Lemma 5.8. *Let $(i_1, i_2), (j_1, j_2) \in \mathbb{Z}_2^2$ with $i_2 = 1$ or $j_2 = 1$, and let $\alpha, \beta \in [0, \pi)$. Then $\mathbb{O}_{G_{\alpha, 0}^{j_1, j_2}, G_{\beta, 0}^{i_1, i_2}} \in \mathcal{D}_{1,1,3,3}$ if and only if*

$$(\cos 2\alpha, \cos 2\beta) \neq ((-1)^{j_1+j_2}, (-1)^{i_1+i_2}).$$

Proof. Applying Proposition 25 in [19] and its proof to the present setting implies the “if”-part of the statement, while the converse is established in Examples 22–24 in [19]. \square

Note that for each $(i_1, i_2), (j_1, j_2) \in \mathbb{Z}_2^2$ with $i_2 = 1$ or $j_2 = 1$, there is exactly one pair $(\alpha, \beta) \in [0, \pi)^2$ such that $\mathbb{O}_{G_{\alpha, 0}^{j_1, j_2}, G_{\beta, 0}^{i_1, i_2}} \notin \mathcal{D}_{1,1,3,3}$.

The reason for taking the seemingly complicated detour via algebras of the form $\mathbb{O}_{F_\xi^{j_1, j_2}, F_\eta^{i_1, i_2}}$ is that isomorphisms between these algebras are easier to construct explicitly. The following remark provides the background.

Remark 5.9. If $A = \mathbb{O}_{F_\xi^{j_1, j_2}, F_\eta^{i_1, i_2}} \in \mathcal{D}_{1,1,3,3}$, then $A_0 = \mathbb{C}$ as a vector subspace. Indeed, we know from [19] that this is the case for the algebras in $\mathcal{D}_{1,1,3,3}$ of the form $\mathbb{O}_{G_{\alpha, \gamma}^{j_1, j_2}, G_{\beta, \gamma}^{i_1, i_2}}$, and the isomorphisms in the proof of Proposition 5.6 all map \mathbb{C} to itself. Thus $A_0 \in \mathcal{A}_2$, and by construction, the double sign of A_0 is $((-1)^{i_1}, (-1)^{j_1})$. It is known that each algebra in \mathcal{A}_2 has precisely three non-zero idempotents if its double sign is $(-, -)$, and a unique non-zero idempotent otherwise. The set of all non-zero idempotents of A_0 is $\{1, (-1 \pm \sqrt{3}u)/2\}$ if $(i_1, j_1) = (1, 1)$, while 1 is the unique non-zero idempotent of A_0 otherwise.

We will solve the classification problem of $\mathcal{D}_{1,1,3,3}$ as follows. Let $\alpha, \beta \in [0, \pi)$ and set $A = \mathbb{O}_{G_{\alpha, 0}^{j_1, j_2}, G_{\beta, 0}^{i_1, i_2}}$ and $B = \mathbb{O}_{G_{\alpha', 0}^{j_1, j_2}, G_{\beta', 0}^{i_1, i_2}}$. Proposition 5.6 and its proof provide us with $\xi, \eta, \xi', \eta' \in \mathbb{R}$ and thence algebras $C = \mathbb{O}_{F_\xi^{j_1, j_2}, F_\eta^{i_1, i_2}}$ and $D = \mathbb{O}_{F_{\xi'}^{j_1', j_2'}, F_{\eta'}^{i_1', i_2'}}$ such that $A \simeq C$ and $B \simeq D$. With Proposition 5.10 below we determine all isomorphisms from C to D . Using this we express, in Lemma 5.11, the condition $C \simeq D$ in terms of ξ, η, ξ' and η' . Using 5.6, we express this as an isomorphism condition on A and B in terms of α, β, α' and β' , which enables us to produce a classification in Theorem 5.12.

Proposition 5.10. *Let $(i_1, i_2), (j_1, j_2) \in \mathbb{Z}_2^2$ with $i_2 = 1$ or $j_2 = 1$ and let $\xi, \eta, \xi', \eta' \in \mathbb{R}$. If $C = \mathbb{O}_{F_\xi^{j_1, j_2}, F_\eta^{i_1, i_2}}$ and $D = \mathbb{O}_{F_{\xi'}^{j_1', j_2'}, F_{\eta'}^{i_1', i_2'}}$ are in $\mathcal{D}_{1,1,3,3}$, and $\phi : C \rightarrow D$ is an isomorphism, then $(i_1, j_1, i_2, j_2) = (i_1', j_1', i_2', j_2')$, and $\phi = B_c \rho$ for some $\rho \in G_2^{(u)}$ and $c \in \mathbb{C}$ satisfying*

- (i) $c^3 = 1$ if $i_1 = j_1 = 1$, and
- (ii) $c = 1$, otherwise.

Note that if $\phi = B_c \rho$ with $\rho \in G_2$, then $(L_c \rho, R_c \rho)$ is a pair of triality components of ϕ . By virtue of Proposition 2.3, the above proposition implies that $\phi : C \rightarrow D$ is an isomorphism if and only if $(i_1, j_1, i_2, j_2) = (i_1', j_1', i_2', j_2')$, $\phi = B_c \rho$ with ρ and c as in the proposition, and

$$(5.4) \quad L_c \rho F_\xi^{j_1, j_2} \rho^{-1} B_{\bar{c}} = F_{\xi'}^{j_1, j_2} \quad \text{and} \quad R_c \rho F_\eta^{i_1, i_2} \rho^{-1} B_{\bar{c}} = F_{\eta'}^{i_1, i_2}.$$

(As far as Proposition 2.3 is concerned, the right hand sides should read $(-1)^k F_{\xi'}^{j_1, j_2}$ and $(-1)^k F_{\eta'}^{i_1, i_2}$ for some $k \in \mathbb{Z}_2$; however, if $k = 1$, then applying either equation to 1 gives $c = -1$ or $c^3 = -1$, depending on (i_1, j_1, i_2, j_2) . Thus necessarily $k = 0$.)

Proof. Invariance of the double sign implies that $(i_1 + i_2, j_1 + j_2) = (i_1' + i_2', j_1' + j_2')$. Since ϕ induces an isomorphism from C_0 to D_0 , Remark 5.9 gives $(i_1, j_1) = (i_1', j_1')$, proving the first part of the statement. Since $C_0 = D_0 = \mathbb{C}$ as subspaces, we have $\phi(\mathbb{C}) = \mathbb{C}$. If $(i_1, j_1) \neq (1, 1)$, then Remark 5.9 implies that $\phi(1) = 1$. Remark 2.5 then implies that $\phi \in G_2$, and by orthogonality, $\phi(\langle u \rangle) = \langle u \rangle$, whence $\phi \in G_2^{(u)}$. This proves the second part in case $(i_1, j_1) \neq (1, 1)$. If $(i_1, j_1) = (1, 1)$, then by Proposition 2.3 we have

$$\phi_1 F_\alpha^{1,1} = F_{\alpha'}^{1,1} \phi \quad \text{and} \quad \phi_2 F_\beta^{1,1} = F_{\beta'}^{1,1} \phi$$

for some pair (ϕ_1, ϕ_2) of triality components of ϕ . Applying both equations to 1, we get $\phi_1(1) = \phi_2(1) = \overline{\phi(1)}$. Then $\phi_1 = R_{\overline{\phi(1)}}\phi$ and $\phi_2 = L_{\overline{\phi(1)}}\phi$ give

$$\phi_1 = R_{\phi(1)}\phi \quad \text{and} \quad \phi_2 = L_{\phi(1)}\phi.$$

By Remark 5.9 we know that $\phi(1) = e$ for some $e \in \mathbb{C}$ satisfying $e^3 = 1$ with respect to the multiplication in \mathbb{O} . Set $x = \phi^{-1}(v)$ and $y = \phi^{-1}(z)$. Then

$$\phi(xy) = \phi_1(x)\phi_2(y) = (ve)(ez) = (\bar{e}v)(z\bar{e}) = \bar{e}((vz)\bar{e}) = \bar{e}(e(vz)) = vz,$$

where the third and fifth equalities hold since $uv' = -v'u$ for each $v' \in \mathbb{C}^\perp$, and the fourth and last equalities follow from alternativity of \mathbb{O} . Since ϕ maps \mathbb{C}^\perp to itself, we in particular have $xy \perp u$. Thus (u, x, y) is a Cayley triple, and there exists $\rho' \in G_2^{(u)}$ mapping (u, v, z) to (u, x, y) . Moreover, $(x, ux, y, uy, xy, (ux)y)$ is a basis of \mathbb{C}^\perp . Now $e = \cos(2\pi n/3) + u \sin(2\pi n/3)$ for some $n \in \mathbb{Z}$, and then $\phi(u) = (-1)^\varepsilon(u \cos(2\pi n/3) - \sin(2\pi n/3))$ for some $\varepsilon \in \mathbb{Z}_2$. By computations similar to those used to determine $\phi(xy)$ we can determine the image of ux, uy and $(ux)y$ in terms of u, v and z . As a result we see that ϕ maps

$$(x, ux, y, uy, xy, (ux)y) \quad \text{to} \quad (v, (-1)^\varepsilon uv, z, (-1)^\varepsilon uz, vz, (-1)^\varepsilon (uv)z).$$

The map $\widehat{\varepsilon} \in G_2^{(u)}$ was defined by $(u, v, z) \mapsto ((-1)^\varepsilon u, v, z)$. By the above, $\phi\rho'\widehat{\varepsilon}$ acts as identity on \mathbb{C}^\perp , and the matrix of its restriction to \mathbb{C} in the basis $(1, u)$ is $(2\pi n/3)_0$. The same is true for B_d with $d = \cos(\pi n/3) + u \sin(\pi n/3)$. Since $B_d = B_{-d}$, we have $\phi\rho'\widehat{\varepsilon} = B_c$ for some $c \in \mathbb{C}$ with $c^3 = 1$. The proof is complete upon observing that $\rho = (\rho'\widehat{\varepsilon})^{-1} \in G_2^{(u)}$. \square

In particular, this shows that the twelve families determined by the values of i_1, i_2, j_1 and j_2 are closed under isomorphisms. Since $A \simeq B$ implies $A_0 \simeq B_0$, we have

$$\mathcal{D}_{1,1,3,3} = \coprod_{\substack{i_1, j_1, i_2, j_2 \in \mathbb{Z}_2 \\ i_2=1 \vee j_2=1}} \mathcal{D}_{1,1,3,3}^{i_1, j_1, i_2, j_2},$$

where $\mathcal{D}_{1,1,3,3}^{i_1, j_1, i_2, j_2}$ is the full subcategory of all $A \in \mathcal{D}_{1,1,3,3}$ with double sign $((-1)^{i_1+i_2}, (-1)^{j_1+j_2})$ and such that the double sign of A_0 is $((-1)^{i_1}, (-1)^{j_1})$. Thus classifying $\mathcal{D}_{1,1,3,3}$ amounts to classifying each block in the above decomposition.

The next lemma expresses the condition that (5.4) holds for some ρ and c , satisfying the conditions of Proposition 5.10, in terms of ξ, η, ξ' and η' .

Lemma 5.11. *Let $(i_1, i_2), (j_1, j_2) \in \mathbb{Z}_2^2$ with $i_2 = 1$ or $j_2 = 1$ and let $\xi, \eta, \xi', \eta' \in \mathbb{R}$.*

- (i) *If $(i_1, j_1) \neq (1, 1)$, then (5.4) holds for some $\rho \in G_2^{(u)}$ and with $c = 1$ if and only if there exist $m, n \in \mathbb{Z}$ such that*

$$(\xi', \eta') = \begin{cases} \pm(\xi + \pi m/3, \eta + \pi n) & \text{if } (i_2, j_2) = (0, 1), \\ \pm(\xi + \pi m, \eta + \pi n/3) & \text{if } (i_2, j_2) = (1, 0), \\ \pm(\xi + \pi m/3, \eta + (m + 3n)\pi/3) & \text{if } (i_2, j_2) = (1, 1). \end{cases}$$

- (ii) *If $(i_1, j_1) = (1, 1)$, then (5.4) holds for some $\rho \in G_2^{(u)}$ and $c \in \mathbb{C}$ with $c^3 = 1$ if and only if there exist $k, m, n \in \mathbb{Z}$ such that*

$$(\xi', \eta') = \begin{cases} \pm(\xi + \pi m/3, \eta + \pi n/3) & \text{if } (i_2, j_2) \neq (1, 1), \\ \pm(\xi + \pi m/3, \eta + (2k + m + 3n)\pi/3) & \text{if } (i_2, j_2) = (1, 1). \end{cases}$$

Proof. Let $(i_1, i_2), (j_1, j_2) \in \mathbb{Z}_2^2$ with $i_2 = 1$ or $j_2 = 1$. For any $c \in \mathbb{C}$ satisfying (i) and (ii) of Lemma 5.10 and any $\rho \in G_2^{(u)}$, (5.4) asserts that

$$L_c \rho C_x \sigma_u^{j_1} \sigma_W^{j_2} \rho^{-1} B_{\bar{c}} = C_{x'} \sigma_u^{j_1} \sigma_W^{j_2} \quad \wedge \quad R_c \rho C_y \sigma_u^{i_1} \sigma_W^{i_2} \rho^{-1} B_{\bar{c}} = C_{y'} \sigma_u^{i_1} \sigma_W^{i_2}$$

where $x = \cos \xi + u \sin \xi$, $y = \cos \eta + u \sin \eta$, and likewise for x', y', ξ' and η' . Since $\rho C_t \rho^{-1} = C_{\rho(t)}$ and $\sigma_W B_t = B_t \sigma_W$ for each $t \in \langle 1, u \rangle$, this is equivalent to

$$C_{\rho(x)} \rho \sigma_u^{j_1} \sigma_W^{j_2} \rho^{-1} = L_{\bar{c}} C_{x'} \sigma_u^{j_1} B_c \sigma_W^{j_2} \quad \wedge \quad C_{\rho(y)} \rho \sigma_u^{i_1} \sigma_W^{i_2} \rho^{-1} = R_{\bar{c}} C_{y'} \sigma_u^{i_1} B_c \sigma_W^{i_2}.$$

If $j_1 = i_1 = 1$, then $\sigma_u^{j_1} B_c = B_{\bar{c}} \sigma_u^{j_1}$ and $\sigma_u^{i_1} B_c = B_{\bar{c}} \sigma_u^{i_1}$ since $\sigma_u B_c = B_{\bar{c}} \sigma_u$. In the other cases, this holds trivially as then $c = 1$ implies that $B_c = B_{\bar{c}} = \text{Id}$. Moreover, σ_u commutes with both σ_W and ρ^{-1} . Multiplying the equations from the right by $\sigma_u^{j_1+j_2}$ and $\sigma_u^{i_1+i_2}$, respectively, and writing Σ for $\sigma_u \sigma_W$, we therefore have

$$C_{\rho(x)} \rho \Sigma^{j_2} \rho^{-1} = L_{\bar{c}} C_{x'} B_{\bar{c}} \Sigma^{j_2} \quad \wedge \quad C_{\rho(y)} \rho \Sigma^{i_2} \rho^{-1} = R_{\bar{c}} C_{y'} B_{\bar{c}} \Sigma^{i_2}.$$

We finally collect the multiplication operators in each equation in the left hand side, and multiply them. Using commutativity of the subalgebra \mathbb{C} of \mathbb{O} , and the fact that $c^3 = 1$, it is straight-forward to check that the above is equivalent to

$$(5.5) \quad C_{\bar{c}x'} \rho(x) = \Sigma^{j_2} \rho \Sigma^{j_2} \rho^{-1} \quad \wedge \quad C_{\bar{c}y'} \rho(y) = \Sigma^{i_2} \rho \Sigma^{i_2} \rho^{-1}.$$

The right hand sides both belong to G_2 , since for each $k \in \mathbb{Z}_2$, Σ^k coincides with $\widehat{k} : (u, v, z) \mapsto ((-1)^k u, v, z)$. If $t \in \mathbb{S}^1$, then clearly $C_t = \text{Id}$ if and only if $t^2 = 1$, while $C_t = \Sigma \rho \Sigma \rho^{-1}$ for some $\rho \in G_2^{(u)}$ if and only if $t^6 = 1$. (To see this, assume that $t^6 = 1$. Then $(rt)^3 = 1$ for some $r \in \{\pm 1\}$, and $C_t = C_{rt}$. Taking $s \in \mathbb{C}$ with $s^2 = rt$ we get $C_t = \Sigma C_s \Sigma C_s^{-1}$. Since $s^6 = 1$, we can apply the fact that $w \in \mathbb{S}^1$ satisfies $C_w \in G_2^{(u)}$ if and only if $w^6 = 1$. Conversely, if $t^6 \neq 1$, then this fact implies that $C_t \notin G_2^{(u)}$, and in particular $C_t \neq \Sigma \rho \Sigma \rho^{-1}$ for any $\rho \in G_2^{(u)}$.)

Returning to (5.5), we notice that $(\rho(x), \rho(y)) \in \{(x, y), (\bar{x}, \bar{y})\}$ for any $\rho \in G_2^{(u)}$. Note also that the right hand sides coincide whenever $(i_2, j_2) = (1, 1)$. Altogether this gives that (5.5) holds for *some* $\rho \in G_2^{(u)}$ and with $c = 1$ if and only if

$$\begin{aligned} \begin{pmatrix} x'^6, y'^2 \end{pmatrix} &= (x^6, y^2) \vee \begin{pmatrix} x'^6, y'^2 \end{pmatrix} = (\bar{x}^6, \bar{y}^2) & \text{if } (i_2, j_2) = (0, 1), \\ \begin{pmatrix} x'^2, y'^6 \end{pmatrix} &= (x^2, y^6) \vee \begin{pmatrix} x'^2, y'^6 \end{pmatrix} = (\bar{x}^2, \bar{y}^6) & \text{if } (i_2, j_2) = (1, 0), \\ \begin{pmatrix} x'^6, (x' \bar{y}')^2 \end{pmatrix} &= (x^6, (x \bar{y})^2) \vee \begin{pmatrix} x'^6, (x' \bar{y}')^2 \end{pmatrix} = (\bar{x}^6, (\bar{x} \bar{y})^2) & \text{if } (i_2, j_2) = (1, 1), \end{aligned}$$

while (5.5) holds for *some* $\rho \in G_2^{(u)}$ and *some* $c \in \mathbb{C}$ with $c^3 = 1$ if and only if

$$\begin{aligned} \begin{pmatrix} x'^6, y'^6 \end{pmatrix} &= (x^6, y^6) \vee \begin{pmatrix} x'^6, y'^6 \end{pmatrix} = (\bar{x}^6, \bar{y}^6) & \text{if } (i_2, j_2) \neq (1, 1), \\ \begin{pmatrix} x'^6, (x' \bar{y}')^6 \end{pmatrix} &= (x^6, (x \bar{y})^6) \vee \begin{pmatrix} x'^6, (x' \bar{y}')^6 \end{pmatrix} = (\bar{x}^6, (\bar{x} \bar{y})^6) & \text{if } (i_2, j_2) = (1, 1). \end{aligned}$$

Writing these conditions in terms of ξ, η, ξ' and η' completes the proof. \square

Expressing these conditions in terms of the parameters α and β yields a classification of each $\mathcal{D}_{1,1,3,3}^{i_1, j_1, i_2, j_2}$ as follows.

Theorem 5.12. *Let $(i_1, i_2), (j_1, j_2) \in \mathbb{Z}_2^2$ with $i_2 = 1$ or $j_2 = 1$. A classification of $\mathcal{D}_{1,1,3,3}^{i_1, j_1, i_2, j_2}$ is given by*

$$M = \left\{ \mathbb{O}_{G_{\alpha,0}^{j_1, j_2}, G_{\beta,0}^{i_1, i_2}} \mid (\alpha, \beta) \in \Pi \setminus \{(j_1 + j_2)\pi/2, (i_1 + i_2)\pi/2\} \right\}$$

where

$$\Pi = ([0, \pi/2) \times [0, \pi)) \cup (\{\pi/2\} \times [0, \pi/2]).$$

Proof. Let $\alpha, \beta, \alpha', \beta' \in [0, \pi)$ and let (ξ, η) and (ξ', η') be obtained from (α, β) and (α', β') , respectively, by equations (5.2) and (5.3), with $\gamma = \gamma' = 0$. In the case $(i_1, j_1, i_2, j_2) = (0, 0, 0, 1)$ we thus have

$$(\xi, \eta) = \left(\frac{\alpha+2\beta}{3}, -\beta \right) \quad \text{and} \quad (\xi', \eta') = \left(\frac{\alpha'+2\beta'}{3}, -\beta' \right)$$

and similar expressions hold for the other cases. Using this, we write the conditions in Lemma 5.11 in terms of (α, β) and (α', β') . Each condition then in fact turns out to be equivalent to

$$(5.6) \quad (\alpha', \beta') = (\alpha, \beta) \quad \vee \quad (\alpha', \beta') = (\pi - \alpha, \pi - \beta).$$

Thus if

$$\mathbb{O}_{G_{\alpha,0}^{j_1,j_2}, G_{\beta,0}^{i_1,i_2}}, \mathbb{O}_{G_{\alpha',0}^{j_1,j_2}, G_{\beta',0}^{i_1,i_2}} \in \mathcal{D}_{1,1,3,3}^{i_1,j_1,i_2,j_2},$$

then they are isomorphic if and only if (5.6) holds, whence M is irredundant. Since each $A \in \mathcal{D}_{1,1,3,3}^{i_1,j_1,i_2,j_2}$ is isomorphic to $\mathbb{O}_{G_{\alpha,0}^{j_1,j_2}, G_{\beta,0}^{i_1,i_2}}$ for some $(\alpha, \beta) \in [0, \pi)^2$ different from $((j_1 + j_2)\pi/2, (i_1 + i_2)\pi/2)$, it follows that M is exhaustive as well. \square

Remark 5.13. The theorem and its proof imply that if

$$A = \mathbb{O}_{G_{\alpha,0}^{j_1,j_2}, G_{\beta,0}^{i_1,i_2}} \simeq \mathbb{O}_{G_{\alpha',0}^{j_1,j_2}, G_{\beta',0}^{i_1,i_2}} = B,$$

then $\widehat{\varepsilon} : A \rightarrow B$ is an isomorphism for some $\varepsilon \in \mathbb{Z}_2$, and conversely. Thus *a posteriori* we get a quasi-description of $\mathcal{D}_{1,1,3,3}^{i_1,j_1,i_2,j_2}$ as follows. Let $j = j_1 + j_2$ and $i = i_1 + i_2$ with $i_2 = 1$ or $j_2 = 1$. The group \mathbb{Z}_2 acts on the set $(\mathbb{S}^1 \times \mathbb{S}^1)_{ij}$ defined in Proposition 3.5 by

$$(5.7) \quad \varepsilon \cdot (a, b) = (K^\varepsilon(a), K^\varepsilon(b)),$$

and the map

$$\mathbb{Z}_2(\mathbb{S}^1 \times \mathbb{S}^1)_{ij} \rightarrow \mathcal{D}_{1,1,3,3}^{i_1,j_1,i_2,j_2}$$

defined on objects by mapping $(a, b) = (\cos 2\alpha + u \sin 2\alpha, \cos 2\beta + u \sin 2\beta)$ with $\alpha, \beta \in [0, \pi)$ to $\mathbb{O}_{G_{\alpha,0}^{j_1,j_2}, G_{\beta,0}^{i_1,i_2}}$, and on morphisms by $\varepsilon \mapsto \widehat{\varepsilon}$, is a dense functor which detects non-isomorphic objects. However, we do not know how, if at all, this can be deduced *a priori*.

Let now $i_2 = j_2 = 0$ and set $i = i_1$ and $j = j_1$. Then the action of \mathbb{Z}_2 on $(\mathbb{S}^1 \times \mathbb{S}^1)_{ij}$ induced by (3.1) coincides with the action defined in (5.7), and it follows from Proposition 3.5 that the restriction of the functor \mathcal{F}^{ij} defined there to $\mathbb{Z}_2(\mathbb{S}^1 \times \mathbb{S}^1)_{ij}$ is a quasi-description of $\mathcal{D}_{1,1,6}^{ij}$. With our observation earlier in this section that $G_{\theta,0}^{k,0} = \lambda_t^{(k)}$, with $t = \cos 2\theta + u \sin 2\theta$, for each $\theta \in \mathbb{R}$ and $k \in \mathbb{Z}_2$, this brings the treatment of $\mathcal{D}_{1,1,6}$ and $\mathcal{D}_{1,1,3,3}$ on an equal footing.

6. CLASSIFICATION OF \mathcal{L}_0 AND \mathcal{H}_0

We will now classify $SO_3(\mathbb{S}^3 \times \mathbb{S}^3)$ and $SO_3[\mathbb{S}^3 \times \mathbb{S}^3]^2$ up to isomorphism, by finding transversals for the actions of SO_3 on $\mathbb{S}^3 \times \mathbb{S}^3$ and $[\mathbb{S}^3 \times \mathbb{S}^3]^2$ defined in Section 4.1. The classification of the first groupoid gives, via the functors \mathcal{J}^{ij} from Section 4.2, a classification of the subcategory $\mathcal{L}_0 \subset \mathcal{D}$. The classification of the second becomes, after removing a finite set of objects, a classification of the

category $_{SO_3}S$ from Section 4.3, which in turn gives a classification of $\mathcal{H}_0 \subset \mathcal{D}$ via the functors \mathcal{K}_*^{ij} introduced in that section.

Recall that we identify $\mathbb{S}(\mathbb{H})$ with \mathbb{S}^3 , and fix an orthonormal pair (u, v) in $\mathbb{H} \cap 1^\perp$. We now define the sets

$$P = \{\cos \alpha + (u \cos \beta + v \sin \beta) \sin \alpha \mid \alpha \in (0, \pi) \wedge \beta \in [0, \pi]\},$$

and

$$P_0 = \{\cos \alpha + u \sin \alpha \mid \alpha \in (0, \pi)\},$$

and, for each $A \subseteq (0, \pi)$ and $B \subseteq [0, \pi]$, we set

$$P_{AB} = \{\cos \alpha + (u \cos \beta + v \sin \beta) \sin \alpha \mid \alpha \in A \wedge \beta \in B\}.$$

Proposition 6.1. *A transversal for the action of SO_3 on $\mathbb{S}^3 \times \mathbb{S}^3$ is given by*

$$M = (\{\pm 1\} \times \{\pm 1\}) \cup (\{\pm 1\} \times P_0) \cup (P_0 \times \{\pm 1\}) \cup (P_0 \times P).$$

Proof. First we show that M exhausts the orbits. Take any $(a, b) \in \mathbb{S}^3 \times \mathbb{S}^3$. If $a \in \langle 1 \rangle$, then $(a, b) = (\pm 1, \cos \alpha + w \sin \alpha)$ for some $\alpha \in [0, \pi]$ and $w \in \mathbb{S}(1^\perp)$. Since SO_3 acts transitively on $\mathbb{S}(1^\perp)$, there exists $q \in \mathbb{S}^3$ such that $qw\bar{q} = u$, and then

$$\kappa_q \cdot (a, b) \in (\{\pm 1\} \times \{\pm 1\}) \cup (\{\pm 1\} \times P_0).$$

If $a \notin \langle 1 \rangle$, then $a = \cos \alpha + w \sin \alpha$ for some $\alpha \in (0, \pi)$ and $w \in \mathbb{S}(1^\perp)$. As above there exists $q \in \mathbb{S}^3$ such that $\kappa_q \cdot (a, b) = (\cos \alpha + u \sin \alpha, b')$, for some $b' \in \mathbb{S}^3$. Then $b' = \cos \alpha' + (u \cos \beta' + w' \sin \beta') \sin \alpha'$ for some $\alpha' \in (0, \pi)$, $\beta' \in [0, \pi]$ and $w' \in \mathbb{S}(\langle 1, u \rangle^\perp)$. Moreover, the subgroup of SO_3 consisting of all elements fixing u acts transitively on $\langle v, uv \rangle$ and thus there exists $q' \in \mathbb{S}^3$ such that $q'w'\bar{q}' = v$. Altogether $\kappa_{q'q} \cdot (a, b) \in (P_0 \times \{\pm 1\}) \cup (P_0 \times P)$. Thus M is exhaustive. To prove irredundancy, we recall that conjugation by a unit quaternion fixes $1 \in \mathbb{H}$. Thus if $\alpha, \alpha' \in [0, \pi]$ satisfy $\kappa_q(\cos \alpha + u \sin \alpha) = \cos \alpha' + u \sin \alpha'$ for some $q \in \mathbb{S}^3$, then $\alpha = \alpha'$. This implies that no orbit intersects more than one of the four sets $\{\pm 1\} \times \{\pm 1\}$, $\{\pm 1\} \times P_0$, $P_0 \times \{\pm 1\}$ and $P_0 \times P$, and that the first three of these sets are irredundant. It moreover proves that if $\kappa_q \cdot (a, b) = (a', b')$ for some $q \in \mathbb{S}^3$ and $(a, b), (a', b') \in P_0 \times P$, then $\kappa_q(a) = a'$ gives $a = a'$ and $\kappa_q(u) = u$, and by coordinatewise comparison of b and b' one concludes that they coincide. Thus $P_0 \times P$ is irredundant. \square

Thus we have classified $_{SO_3}(\mathbb{S}^3 \times \mathbb{S}^3)$, and a classification of \mathcal{L}_0^{ij} is obtained for each $(i, j) \in \mathbb{Z}_2^2$ by applying the functor \mathcal{J}^{ij} of Theorem 4.6 to this classification.

In order to classify $_{SO_3}[\mathbb{S}^3 \times \mathbb{S}^3]^2$, we will first obtain a transversal for the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]$, and pass to the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]^2$. The strategy is detailed in the following lemma, for which we set

$$(6.1) \quad M_1 = (\{1\} \times \{\pm 1\}) \cup (\{1\} \times P_0) \cup (P_0 \times \{1\}) \cup (P_0 \times P)^+,$$

with

$$(P_0 \times P)^+ = \{(a, b) \in P_0 \times P \mid (\Re(a), \Re(b)) \succeq (0, 0)\},$$

where \succeq is the relation “greater than or equal to” with respect to the lexicographic order on \mathbb{R}^2 , and \Re denotes the orthogonal projection onto $\langle 1 \rangle$.

Lemma 6.2. *Let M_1 be as in (6.1).*

- (i) *The quotient map $\mathbb{S}^3 \times \mathbb{S}^3 \rightarrow [\mathbb{S}^3 \times \mathbb{S}^3]$ maps M_1 bijectively onto $[M_1]$, and $[M_1]$ is a transversal of the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]$.*

- (ii) The set $[M_1] \times [\mathbb{S}^3 \times \mathbb{S}^3]$ exhausts the orbits of the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]^2$.
- (iii) Two elements $([a_1, b_1], [a_2, b_2])$ and $([c_1, d_1], [c_2, d_2])$ in $[M_1] \times [\mathbb{S}^3 \times \mathbb{S}^3]$ are in the same orbit if and only if $[a_1, b_1] = [c_1, d_1]$ and $\kappa_q \cdot [a_2, b_2] = [c_2, d_2]$ for some $q \in \mathbb{S}^3$ with $\kappa_q \in \text{Stab}([a_1, b_1])$.

Proof. The bijectivity statement in (i) follows from the fact that M_1 does not contain the negative of any of its elements. To show that $[M_1]$ is exhaustive, we note that Proposition 6.1 implies that $[M]$ indeed is. Let thus $(a, b) \in M$. By construction, M_1 contains either (a, b) or $(-\bar{a}, -\bar{b})$. Since SO_3 acts transitively on $\mathbb{S}(1^\perp)$, the elements $[a, b]$ and $[\bar{a}, \bar{b}] = [-\bar{a}, -\bar{b}]$ are in the same orbit of the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]$. Thus $[M_1]$ is exhaustive.

To show that it is irredundant, assume that for some (a, b) and (c, d) in M_1 , $[a, b]$ and $[c, d]$ are in the same orbit. Then there exist $\varepsilon \in \mathbb{Z}_2$ and $q \in \mathbb{S}^3$ such that $(\kappa_q(a), \kappa_q(b)) = ((-1)^\varepsilon c, (-1)^\varepsilon d)$. If one of a, b, c and d is not orthogonal to 1, then either $\Re(a)$ and $\Re(c)$ are positive, or $\Re(b)$ and $\Re(d)$ are positive. In both cases $\varepsilon = 0$, which implies that $(a, b) = (c, d)$ since $M_1 \subset M$ is irredundant with respect to the action of SO_3 on $\mathbb{S}^3 \times \mathbb{S}^3$. If all of a, b, c and d are orthogonal to 1, then $(a, b) = (u, u \cos \beta + v \sin \beta)$ and $(c, d) = (u, u \cos \beta' + v \sin \beta')$, for some $\beta, \beta' \in [0, \pi]$. Then $(\kappa_q(a), \kappa_q(b)) = ((-1)^\varepsilon c, (-1)^\varepsilon d)$ implies that

$$\kappa_q(u) = (-1)^\varepsilon u \quad \text{and} \quad \kappa_q(u) \cos \beta + \kappa_q(v) \sin \beta = (-1)^\varepsilon u \cos \beta' + (-1)^\varepsilon v \sin \beta',$$

and substituting the first equation into the second gives $\cos \beta = \cos \beta'$, whence $(a, b) = (c, d)$. This proves irredundancy and completes the proof of (i), and (ii) and (iii) follow. \square

In view of the above lemma, two tasks remain in order to obtain a transversal for the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]^2$. The first is to determine the stabilizers of the elements in $[M_1]$ with respect to the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]$. Secondly, we must compute, for each such stabilizer G , a transversal for the restriction to $G \leq SO_3$ of the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]$. These tasks are carried out in the following two lemmata.

Lemma 6.3. *Let M_1 be as in (6.1), and let $(a, b) \in M_1$.*

- (i) *If $a, b \in \langle 1 \rangle$, then $\text{Stab}([a, b]) = SO_3$.*
- (ii) *If $a, b \in \langle 1, u \rangle$ and neither both belong to $\langle 1 \rangle$ nor both belong to $\langle u \rangle$, then $\text{Stab}([a, b]) = \{\kappa_q | q \in \mathbb{S}(\langle 1, u \rangle)\}$.*
- (iii) *If $a, b \in \langle u \rangle$, then $\text{Stab}([a, b]) = \{\kappa_q | q \in \mathbb{S}(\langle 1, u \rangle \cup \langle v, uv \rangle)\}$.*
- (iv) *If $a, b \in \langle u, v \rangle$ and are linearly independent, then $\text{Stab}([a, b]) = \{\text{Id}, \kappa_{uv}\}$.*
- (v) *Otherwise, $\text{Stab}([a, b])$ is trivial.*

Proof. By definition of the action, $\kappa_q \in \text{Stab}([a, b])$ if and only if

$$(\kappa_q(a), \kappa_q(b)) = ((-1)^\varepsilon a, (-1)^\varepsilon b)$$

for some $\varepsilon \in \mathbb{Z}_2$, i.e. if and only if q commutes with both a and b , or anticommutes with them. For each $w \in \mathbb{H}$, the centralizer of w is \mathbb{H} if $w \in \langle 1 \rangle = Z(\mathbb{H})$, and $\langle 1, w \rangle$ otherwise, while the set of all quaternions anticommuting with w is empty if $\Re(w) \neq 0$, and $\langle 1, w \rangle^\perp$ otherwise. The assertions follow. \square

For the next lemma, we introduce, for any $m, n \in \{1, 2, 3, 4\}$, the notation

$$\mathbb{S}_{mn}^3 = \{r_1 + r_2 u + r_3 v + r_4 uv \in \mathbb{S}^3 | (r_m, r_n) \succeq (0, 0)\},$$

where \succeq refers to the lexicographic order.

Lemma 6.4. *Let M_1 be as in (6.1).*

- (i) *A transversal for the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]$ is given by $[M_1]$.*
- (ii) *A transversal for the restriction of the action to $\{\kappa_q | q \in \mathbb{S}(\langle 1, u \rangle)\}$ is given by $[M_2]$, where M_2 is the union of*

$$(\{1\} \cup P_0) \times (\{\pm 1\} \cup P), \quad \{v\} \times \mathbb{S}_{12}^3, \quad \text{and} \quad \left(P_{(0, \frac{\pi}{2})(0, \pi)} \cup P_{\{\frac{\pi}{2}\}(0, \frac{\pi}{2})} \right) \times \mathbb{S}^3.$$

- (iii) *A transversal for the restriction of the action to $\{\kappa_q | q \in \mathbb{S}(\langle 1, u \rangle \cup \langle v, uv \rangle)\}$ is given by $[M_3]$, where M_3 is the union of*

$$\{1\} \times \left(\{\pm 1\} \cup P_{(0, \pi)[0, \frac{\pi}{2}]} \right), \quad P_{(0, \frac{\pi}{2})\{0\}} \times (\{\pm 1\} \cup P),$$

$$\{u\} \times \left(\{1\} \cup P_{(0, \frac{\pi}{2})[0, \pi]} \right), \quad \{v\} \times (\mathbb{S}_{14}^3 \cap \mathbb{S}_{24}^3),$$

$$P_{\{\frac{\pi}{2}\}(0, \frac{\pi}{2})} \times \mathbb{S}_{14}^3, \quad P_{(0, \frac{\pi}{2})\{\frac{\pi}{2}\}} \times \mathbb{S}_{24}^3, \quad \text{and} \quad P_{(0, \frac{\pi}{2})(0, \frac{\pi}{2})} \times \mathbb{S}^3.$$

- (iv) *A transversal for the restriction of the action to $\{\text{Id}, \kappa_{uv}\}$ is given by $[M_4]$, where*

$$M_4 = (\mathbb{S}_{1,4}^3 \cap \mathbb{S}_{2,3}^3) \times \mathbb{S}^3.$$

Moreover, for each $1 \leq k \leq 4$, the quotient map $\mathbb{S}^3 \times \mathbb{S}^3 \rightarrow [\mathbb{S}^3 \times \mathbb{S}^3]$ maps M_k bijectively onto $[M_k]$.

Recall that for each $a, b, c, d \in \mathbb{S}^3$, $[a, b]$ and $[c, d]$ belong to the same orbit of the restriction of the action to $G \leq SO_3$ if and only if $(c, d) = ((-1)^\varepsilon \kappa_q(a), (-1)^\varepsilon \kappa_q(b))$ for some $\varepsilon \in \mathbb{Z}_2$ and $q \in \mathbb{S}^3$ with $\kappa_q \in G$.

Proof. Statement (i) follows immediately from Lemma 6.2(i).

For statement (ii) we note that $\{\kappa_q | q \in \mathbb{S}(\langle 1, u \rangle)\}$ fixes 1 and u and acts transitively on $\mathbb{S}(\langle v, uv \rangle)$. Let now $(a, b) \in \mathbb{S}^3 \times \mathbb{S}^3$. If $a \in \langle 1, u \rangle$, then $(-1)^\varepsilon a \in \{1\} \cup P_0$ for some $\varepsilon \in \mathbb{Z}_2$, and there exists $q \in \mathbb{S}(\langle 1, u \rangle)$ such that κ_q maps b into $\{\pm 1\} \cup P$ and fixes a . If $a \notin \langle 1, u \rangle$, then there exists $q \in \mathbb{S}(\langle 1, u \rangle)$ such that the projection of $\kappa_q(a)$ onto $\langle v, uv \rangle$ is $(-1)^\varepsilon v$, where ε is determined as follows. If $a \perp \langle 1, u \rangle$, then ε is chosen so that $(-1)^\varepsilon b \in \mathbb{S}_{1,2}^3$. If not, it can be chosen so that $(-1)^\varepsilon a \in \mathbb{S}_{1,2}^3$. In both cases we get $((-1)^\varepsilon \kappa_q(a), (-1)^\varepsilon \kappa_q(b)) \in M_2$. Altogether we have shown that $[M_2]$ exhausts the orbits of the action.

To prove that $[M_2]$ is irredundant, assume that (a, b) and (c, d) in M_2 satisfy $(c, d) = ((-1)^\varepsilon \kappa_q(a), (-1)^\varepsilon \kappa_q(b))$ for some $\varepsilon \in \mathbb{Z}_2$ and $q \in \mathbb{S}(\langle 1, u \rangle)$. Since conjugation by q fixes 1 and u and maps $\langle v, uv \rangle$ to itself, it follows that (a, b) and (c, d) belong to the same one of the three constituents³ of M_2 . It moreover follows that

$$(6.2) \quad (x, y \in \{\pm 1\} \cup P \wedge \kappa_q(x) = y) \implies x = y.$$

Now if (a, b) and (c, d) lie in $(\{1\} \cup P_0) \times (\{\pm 1\} \cup P)$, then q commutes with a , implying that $\kappa_q(a) = a$, and then $\varepsilon = 0$ as $-a \notin \{1\} \cup P_0$. Then $a = c$, and $b, d \in \{\pm 1\} \cup P$ with $\kappa_q(b) = d$, which by (6.2) implies that $b = d$. If (a, b) and (c, d) lie in $\{v\} \times \mathbb{S}_{1,2}^3$, then $(-1)^\varepsilon \kappa_q(v) = v$, implying that either $\varepsilon = 0$ and $\kappa_q = \text{Id}$ or $\varepsilon = 1$ and $\kappa_q = \kappa_u$. If the projection of b onto $\langle 1, u \rangle$ is non-zero, then the first possibility must hold; if not, then $-\kappa_u(b) = b$. In both cases $(a, b) = (c, d)$. Finally let (a, b) and (c, d) be in the rightmost constituent of M_2 . Then each of a and c either has a positive 1-coordinate, or the 1-coordinate is zero and the u -coordinate

³The *constituents* of each M_k are the subsets of $\mathbb{S}^3 \times \mathbb{S}^3$ as a union of which M_k is explicitly constructed in the above lemma. Thus M_2 has three constituents, and M_3 has seven.

is positive. Since κ_q fixes 1 and u this implies that $\varepsilon = 0$, whence $a = c$ by (6.2). Since a has a non-zero v -coordinate we have $\kappa_q(v) = v$, which, in view of $q \in \langle 1, u \rangle$ implies that $\kappa_q = \text{Id}$. Therefore $(a, b) = (c, d)$, completing the proof of (ii).

As for statement (iii), observe that the group acting here contains as a subgroup that of item (ii), whence M_2 exhausts the orbits. The set M_3 is a subset of M_2 , and by computations similar to those in the previous item one sees that for each $(a, b) \in M_2$, either $(a, b) \in M_3$ or there exists $(c, d) \in M_3$ with $(c, d) = ((-1)^\varepsilon \kappa_q(a), (-1)^\varepsilon \kappa_q(b))$ for some $\varepsilon \in \mathbb{Z}_2$ and $q \in \mathbb{S}(\langle v, uv \rangle)$. To prove that M_3 is irredundant, assume that $(c, d) = ((-1)^\varepsilon \kappa_q(a), (-1)^\varepsilon \kappa_q(b))$ for some $(a, b) \neq (c, d) \in M_3$, $\varepsilon \in \mathbb{Z}_2$ and $q \in \mathbb{S}(\langle 1, u \rangle \cup \langle v, uv \rangle)$. Then $q \in \mathbb{S}(\langle v, uv \rangle)$ as M_3 is a subset of M_2 , and $[M_2]$ is irredundant with respect to the restriction of the action to $\{\kappa_q | q \in \mathbb{S}(\langle 1, u \rangle)\}$, by the previous item. Thus the norm of the projection of a onto each of $\langle 1 \rangle$, $\langle u \rangle$ and $\langle v, uv \rangle$ coincides with that of c , which upon inspection implies that (a, b) and (c, d) belong to the same constituent of M_3 . Checking that no such $(a, b), (c, d)$ exists is then done separately in each constituent, using computations similar to those of the previous item.

Statement (iv) is easy to check. Finally the statement that each M_k is mapped bijectively onto $[M_k]$ follows from the fact that for each k , M_k does not contain the negative of any of its elements. \square

Combining the above two lemmata, we immediately arrive at a transversal for the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]^2$ as follows.

Corollary 6.5. *Let M_1, \dots, M_4 be as in Lemma 6.4. A transversal for the action of SO_3 on $[\mathbb{S}^3 \times \mathbb{S}^3]^2$ is given by the set N of all $([a_1, b_1], [a_2, b_2]) \in [M_1] \times [\mathbb{S}^3 \times \mathbb{S}^3]$ satisfying any of the following mutually exclusive conditions.*

- (i) $a_1, b_1 \in \langle 1 \rangle$ and $(a_2, b_2) \in M_1$.
- (ii) $a_1, b_1 \in \langle 1, u \rangle$ and are neither both in $\langle 1 \rangle$ nor both in $\langle u \rangle$, and $(a_2, b_2) \in M_2$.
- (iii) $a_1, b_1 \in \langle u \rangle$ and $(a_2, b_2) \in M_3$.
- (iv) $a_1, b_1 \in \langle u, v \rangle$ and are linearly independent, and $(a_2, b_2) \in M_4$.
- (v) a_1, b_1 satisfy none of the above, and $a_2, b_2 \in \mathbb{S}^3$.

Remark 6.6. The four elements $([1, \pm 1], [1, \pm 1])$ satisfy the first item above. The set $N' = N \setminus \{([1, \pm 1], [1, \pm 1])\}$ is thus a transversal for the action of SO_3 on the set S from Remark 4.8. By virtue of Theorem 4.9 and using the functors \mathcal{K}_*^{ij} defined therein, a classification of the category \mathcal{H}_0^{ij} is given by $\mathcal{K}_*^{ij}(N')$ for each $(i, j) \in \mathbb{Z}_2^2$.

This concludes the classification of \mathcal{D} .

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